KTH ROYAL INSTITUTE OF TECHNOLOGY

Lecture 8

- Field of values
- **Stable matrices**
- (SVD)

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Lecture 8: Content

- Field of values
- **Stable matrices**
- (SVD)

Parts of Chapter 1, 2, and (3) of "Topics in Matrix Analysis," by R. A. Horn and C. R. Johnson + additional material (e.g., from "Matrices - Methods and Applications" by S. Barnett).

The field of values

The field of values (or the numerical range), $F(A)$, is a set of complex numbers associated with a matrix $A \in M_n$:

$$F(A) \equiv \{x^*Ax : \; x \in \mathbf{C}^n, \; x^*x = 1\}$$

Some Properties

The spectrum $\sigma(A)$ of A is a discrete point set while the field of values can be a continuum. $F(A)$ is always a compact convex set.

Clearly,

- $F(A+B) \subset F(A) + F(B)$
  (sum of all possible element pairs)
- $\sigma(A) \subset F(A)$

Field of values, cont'd

Can be useful to derive conditions on eigenvalues, e.g., in stability analysis.

Example: If $A + A^*$ is positive definite, then A is positive stable (i.e., all its eigenvalues are in the right half plane of $\mathbf{C}$.)

Stable matrices: Introduction

System of n first order differential equations ($A \in M_n$):

$$\frac{dx(t)}{dt} = A(x(t) - \hat{x}); \quad x(0) = x_0$$

Solution: $x(t) = e^{At}(x_0 - \hat{x}) + \hat{x}$

where $e^{At} \equiv \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$

Fact: $\hat{x}$ is a globally stable equilibrium iff the eigenvalues of A have negative real parts.

For nonlinear differential equations we may check local stability by studying the linearized system (sufficient conditions).

Stable matrices: Inertia for complex matrices

Def: Let $A \in M_n$.

$i_+(A)$ is the number (with multiplicities) of eigenvalues of A with positive real part.

$i_-(A)$ is the number (with multiplicities) of eigenvalues of A with negative real part.

$i_0(A)$ is the number (with multiplicities) of eigenvalues of A with zero real part.

Clearly, $i_+(A) + i_-(A) + i_0(A) = n$. We also define the *inertia* of A as

$$i(A) = [i_+(A), i_-(A), i_0(A)]$$

Stable matrices

Def: A matrix $A \in M_n$ is said to be *positive stable* if $i(A) = [n, 0, 0]$.

Clearly, if A is positive stable

- $-A$ is negative stable; and
- A^{-1}, A^*, A^T are positive stable.

Connections: Matrices – Polynomials

- All roots of a polynomial lie in the right half-plane if and only if the corresponding companion matrix is positive stable.
- A matrix is positive stable if and only if the roots of its characteristic polynomial lie in the right half-plane.

Lyapunov's theorem

$A \in M_n$ is positive stable if and only if there exists a positive definite $G \in M_n$ such that

$$GA + A^*G = H \qquad (1)$$

is positive definite.

Furthermore, if there are Hermitian matrices $G, H \in M_n$ that satisfy (1) and suppose H is positive definite, then A is positive stable if and only if G is positive definite.

Lyapunov cont'd

Thm: If A is positive stable, Lyapunov's equation has a unique solution G for each H. If H is Hermitian, so is G and if H is positive definite, then G is positive definite (more later).

Given a positive stable matrix A, the Lyapunov equation may be seen as a function that for each positive definite matrix H produces a (unique) positive definite matrix $G \equiv G_A(H)$.

One common form of the Lyapunov equation is

$$GA + A^*G = I$$

Test for stability: Solve for G (Hermitian) and check if it is positive definite.

Lyapunov: The discrete time case

For the difference equation:

$$x(k+1) = Ax(k)$$

we know it is asymptotically stable if and only if all eigenvalues of A are inside the unit circle.
Construct the Lyapunov function $V(k) = x^*(k)Gx(k)$ where $G > 0$ (positive definite) and study

$$\begin{aligned} V(k+1) - V(k) &= x^*(k+1)Gx(k+1) - x^*(k)Gx(k) \\ &= x^*(k)\big(A^*GA - G\big)x(k) \end{aligned}$$

Hence, if $\big(A^*GA - G\big) < 0$, then $V(k)$ will decay towards zero which implies that $\|x(k)\| \to 0$ as $k \to \infty$.

Discrete time Lyapunov theorem

The matrix $A \in M_n$ is convergent ($|\lambda_i(A)| < 1$) if and only if there exist positive definite matrices $G, H \in M_n$ such that

$$A^*GA - G = -H$$

Lyapunov: Generalization

Def: A matrix $A \in M_n$ is *positive semistable* if $i_-(A) = 0$.

Thm: If $GA + A^*G = H$ for some positive definite G and some positive semidefinite H, then A is positive semistable.

Lyapunov theorem cont'd

If H in the Lyapunov equations

$$GA + A^*G = H$$
$$A^*GA - G = -H$$

is only positive semidefinite, then A is positive stable (or convergent, respectively) if and only if the following controllability condition holds:

$$\operatorname{rank}\begin{bmatrix} H & A^*H & (A^*)^2H & \dots & (A^*)^{n-1}H \end{bmatrix} = n$$

The general inertia theorem

Let $A \in M_n$ be given. There exists a Hermitian $G \in M_n$ and a positive definite $H \in M_n$ such that

$$GA + A^*G = H$$

if and and only if $i_0(A) = 0$. In this event, $i(A) = i(G)$.

Routh-Hurwitz matrix

Let $A \in M_n(\mathbf{R})$ and
$p_A(t) = \det(tI - A) = t^n + a_1 t^{n-1} + \dots + a_n$.
Construct the Routh-Hurwitz matrix:

$$\Omega(A) = \begin{bmatrix} a_1 & a_3 & a_5 & \dots & a_{2n-1} \\ 1 & a_2 & a_4 & \dots & a_{2n-2} \\ 0 & a_1 & a_3 & \dots & a_{2n-3} \\ 0 & 1 & a_2 & \dots & a_{2n-4} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}$$

where $a_r = 0,\ r > n$.

Routh-Hurwitz conditions

Thm: The matrix A is negative stable if and only if the leading principal minors of $\Omega(A)$ are positive.

Routh-Hurwitz, cont'd

One can show that the Routh-Hurwitz test is equivalent to checking the signs of the diagonal elements in a triangularized version of the Routh-Hurwitz matrix (see, e.g., Barnett "Matrices: Methods and Applications"). This leads to the so called Routh array:

Routh array

Define ($a_0 = 1$):

$$\{r_{01}, r_{02}, r_{03}, \dots\} = \{a_0, a_2, a_4, \dots\}$$
$$\{r_{11}, r_{12}, r_{13}, \dots\} = \{a_1, a_3, a_5, \dots\}$$

Each subsequent row is then obtained from the preceding two by:

$$r_{ij} = \frac{-\det \begin{bmatrix} r_{i-2,1} & r_{i-2,j+1} \\ r_{i-1,1} & r_{i-1,j+1} \end{bmatrix}}{r_{i-1,1}}; \qquad i = 2, 3, \dots, n$$

Routh's thm: The polynomial $p_A(t)$ has all roots in the left half plane if and only if all elements r_{i1} in the first column are positive.

Bilinear transformation

We may also use the bilinear transformations

$$\mu = \frac{\lambda + 1}{\lambda - 1} \qquad \lambda = \frac{\mu + 1}{\mu - 1}$$

which is a one-to-one mapping between the left half of the λ-plane to the unit disc $|\mu| < 1$.
By this we have

$$a(\lambda) = a\left(\frac{\mu + 1}{\mu - 1}\right) = \frac{b(\mu)}{(\mu - 1)^n}$$

and, hence, to check whether $b(\mu)$ has roots inside the unit circle is equivalent to check if $a(\lambda)$ has roots in the left half plane or vice versa.

Chapter 3: SVD

Chapter 3 begins with a review of the singular value decomposition partially overlapping with the first book. It may be fun to read the introductory section about historical remarks. For the ones working with the SVD or whenever you need properties (bounds) on singular values, this is a good source of information.