KTH, Matematik, Maurice Duits

SF2795 Fourier Analysis
Homework Assignment for Lecture 13

1. (3.1.4) Show that $\hat{f}(\gamma)$ is analytic in the whole complex plane if $f \in \mathbb{L}^2(\mathbb{R})$ has compact support, whereas it is analytic in the open strip $|2\pi \operatorname{Im} \gamma| < B$ is $|f(x)| \leq cst * \mathrm{e}^{-B|x|}$.
2. (3.1.13) EXTENDED
   (a) Check that if α belongs to the open upper half plan, then
   $$\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{2\pi \mathrm{i} x\gamma}}{x-\alpha} \mathrm{d}x = \begin{cases} \mathrm{e}^{2\pi\alpha\gamma}, & \gamma > 0 \\ 0, & \gamma < 0 \end{cases}$$
   (b) Use this to check that if $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ then $\hat{f}(\gamma) = \mathrm{e}^{-2\pi|\gamma|}$.
3. (3.1.17) Check that if $B > 0$ and if $|f(x)| \leq \exp(-B|x|)$ for $-\infty \leq a \leq x \leq b \leq \infty$, then the functions $x^n f : n \geq 0$ span $\mathbb{L}^2[a, b]$.
   Hint: Take $g \in \mathbb{L}^2[a, b]$ perpendicular fo $x^n f$ for every $n \geq 0$. Then
   $$(fg^*)^\wedge(\gamma) = \int_a^b fg^* \mathrm{e}^{-2\pi \mathrm{i}\gamma x} \mathrm{d}x$$
   is analytic in the open strip $D : 2\pi|\operatorname{Im} \gamma| < B$ and vanishes together with all its derivatives at $\gamma = 0$.