Homework #4

Numbers below refer to problems in Horn, Johnson "Matrix analysis." A number 1.1.P.2 refers to Problem 2 in Section 1.1.

1. (4.1.P6+P7) Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be given.
   (a) Show that $x^*Ax = x^*Bx$ for all $x \in \mathbb{C}^n$ if and only if $A = B$.
   (b) Show that $x^TAx = 0$ for all $x \in \mathbb{C}^n$ if and only if $A^T = -A$.
   (c) Give an example showing that A and B need not be equal if $x^TAx = x^TBx$ for all $x \in \mathbb{C}^n$.

2. (4.1.P19) Let $A \in M_n$ be a projection ($A^2 = A$). One says that A is a *Hermitian projection* if A is Hermitian, and that A is an *orthogonal projection* if the range of A is orthogonal to its null space. Use the basic properties of Hermitian matrices to show that A is a Hermitian projection if and only if it is an orthogonal projection.

   Hint: $x = (I-A)x+Ax$ is a sum of vectors in the null space and range of A. If the null space is orthogonal to the range, then $x^*Ax = ((I-A)x+Ax))^*Ax = x^(A^*A)x$ is real for all x.

3. Prove that the formulation of Courant-Fischer's max-min theorem shown in the lecture slides (Theorem 4.2.6 in the 2nd edition of the book) is equivalent to

$$\lambda_k = \min_{w_1,\dots,w_{n-k}} \max_{\substack{x\neq 0\\ x\perp w_1,\dots,w_{n-k}}} \frac{x^*Ax}{x^*x}$$

$$\lambda_k = \max_{w_1,\dots,w_{k-1}} \min_{\substack{x\neq 0\\ x\perp w_1,\dots,w_{k-1}}} \frac{x^*Ax}{x^*x}$$

   where $w_i, x \in \mathbf{C}^n$ and the vectors $\{w_i\}$ are allowed to be linearly dependent.

   It is only necessary to prove one of the two expressions above, the other proof will be very similar.

4. Given $A = A^* \in M_n$ and $B = B^* \in M_n$ where B is positive definite.
   (a) show that there is a non-singular matrix X such that $X^*AX = C$ and $X^*BX = D$ where both C and D are diagonal.
   Hint: Write $B = LL^*$ (for example, L can be the Cholesky factor, which we will study in more detail in Lect. 6), apply the spectral factorization on the matrix $L^{-1}AL^{-*}$ and use the result to form X. One of the matrices C and D will end up being the identity matrix.

(b) Given a matrix X such that $X^*AX = C$ and $X^*BX = D$ where both C and D are diagonal (not necessarily obtained using the technique you derived above), show that the columns of X are eigenvectors of the following generalized eigenvalue problem

$$Ax = \lambda Bx$$

and describe how the corresponding eigenvalues can be obtained from C and D.

5. (4.3.P3, 4.3.P7 in the old edition) If $A, B \in M_n$ are Hermitian and their eigenvalues are arranged in nondecreasing order, explain why $\lambda_i(A+B) \leq \min\{\lambda_j(A) + \lambda_k(B) : j + k = i + n\}$.

6. (4.4.P2) Provide details for the following derivation of the Autonne-Takagi factorization, using real valued representations. Let $A \in M_n$ be symmetric. If A is singular and rank $A = r$, it is unitarily congruent to $A' \oplus 0_{n-r}$, in which $A' \in M_r$ is non-singular and symmetric (no need to prove this step). Assume therefore WLOG that A is nonsingular. Let $A = A_1 + iA_2$ with A_1, A_2 real and let $x, y \in \mathbb{R}^n$. Consider the real representation $R_2(A) = \begin{bmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{bmatrix}$, in which A_1, A_2 and $R_2(A)$ are symmetric. Show that

   (a) $R_2(A)$ is nonsingular.

   (b) $R_2(A)\begin{bmatrix} x \\ -y \end{bmatrix} = \lambda \begin{bmatrix} x \\ -y \end{bmatrix}$ if and only if $R_2(A)\begin{bmatrix} x \\ -y \end{bmatrix} = -\lambda \begin{bmatrix} x \\ -y \end{bmatrix}$, so the eigenvalues of $R_2(A)$ appear in $\pm$ pairs.

   (c) Let $\begin{bmatrix} x_1 \\ -y_1 \end{bmatrix}, \ldots, \begin{bmatrix} x_n \\ -y_n \end{bmatrix}$ be orthonormal eigenvectors of $R_2(A)$ associated with its positive eigenvalues $\lambda_1, \ldots, \lambda_n$, let $X = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 & \ldots & y_n \end{bmatrix}$, $\Sigma = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $V = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$ and $\Lambda = \Sigma \oplus (-\Sigma)$. Then V is real orthogonal and $R_2(A) = V\Lambda V^T$. Let $U = X - iY$. Explain why U is unitary and show that $U\Sigma U^T = A$.

7. (a) Let $\alpha = [\alpha_i] \in \mathbf{R}^n$ and $\beta = [\beta_i]$, where $\beta_1 = \cdots = \beta_n = \frac{1}{n}\sum \alpha_i$. Show that α majorizes β.

   (b) [Optional, only the solution to a) is considered in the grading] Let $\Lambda = \text{diag}(\alpha_1, \ldots, \alpha_n)$. Try to find a unitary matrix $U \in M_n$ such that all diagonal elements of $U\Lambda U^*$ are equal. Note that this is a simple special case of Theorem 4.3.48. However, in this special case, it is easy to determine a matrix U that works for all α (in general, U will have to depend on the two vectors).

8. Let $A = A^* \in M_n$ be a positive definite matrix ($\lambda_i(A) > 0$). Show that

$$\log\det(A) - \text{Tr}(A)$$

is maximized by $A = I$.