KTH, Matematik, Maurice Duits

SF2795 Fourier Analysis
Homework Assignment for Lecture 11

1. (2.7.4) Find the temperature in the "semi-infinite" rod $x \geq 0$ if the initial data $u(0+,\cdot) = f$ is known and the left end ($x = 0$) is held at temperature 0, imposing upon f whatever technical conditions you need.
   Answer
$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_0^\infty \big(\exp(-(x-y)^2/2t) - \exp(-(x+y)^2/2t)\big) f(y)\mathrm{d}y.$$

2. (2.8.1) Define for a symmetric linear operator A on $\mathbb{L}^2(\mathbb{R})$ the average
$$\text{average}A = \int A\psi\psi^* = (A\psi, \psi) \qquad (\,= (\psi, A\psi))\,.$$
   Check that
$$\text{average}[A - \text{ave}(A)]^2 \times \text{average}[B - \text{ave}(B)]^2 \geq \frac{1}{4}|\text{average}(AB - BA)|^2,$$
   for any observables (i.e. operator) A and B. Do no worry about domains. A formal proof is all that is asked.
   Hint: The average of $(AB - BA)$ is 2i times the imaginary part of $(B\psi, A\psi)$. Use Schwarz inequality to the check the bound if $\text{ave}(A) = \text{ave}(B) = 0$. Reduce the general case to this one.

3. (2.10.2) Check that the action of the n-th dimensional Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ on radial functions is
$$\Delta = f'' + \frac{n-1}{r} f'.$$

4. (2.10.3). Use the integral formula for j_n in the book to prove that
$$j_n'' + \frac{n-1}{r} j_n' = -j_n.$$
   Hint: use the previous exercise.