

SF2795 Fourier Analysis
Homework Assignment for Lecture 10

1. (Exercise 2.5.7) Prove that $\{h_n\}_{n=0}^\infty$ spans $\mathbb{L}_2(\mathbb{R})$. *Hint:* For any $f \in \mathbb{L}_2(\mathbb{R})$, the Fourier transform of $f \exp(-\pi x^2)$ can be expanded in a power series:

$$\begin{aligned} [f \exp(-\pi x^2)]^\wedge &= \int f(x) \exp(-\pi x^2) e^{-2\pi i \gamma x} dx \\ &= \int f(x) \exp(-\pi x^2) \sum_{n=0}^{\infty} (-2\pi i \gamma)^n (x^n/n!) dx \\ &= \sum_{n=0}^{\infty} (-2\pi i \gamma^n/n!) \int f(x) \exp(-\pi x^2) x^n dx. \end{aligned}$$

Check that this is legitimate, and infer that $f = 0$ is the only function that is perpendicular to Hermite functions. *Amplification:* To do things this way, you have to know that $[f \exp(-\pi x^2)]^\wedge = 0$ only if $f = 0$. Prove this by hand.

Hint:

$$([f \exp(-\pi x^2)]^\wedge, \hat{g}) = (f \exp(-\pi x^2), g)$$

may be proved by hand for any $g \in C_{\downarrow}^\infty(\mathbb{R})$.

2. (Exercise: 2.5.8) Define \hat{f} for the general $f \in \mathbb{L}^2(\mathbb{R})$ by Wiener's recipe:

$$\hat{f} = \sum_{n=0}^{\infty} (f, e_n) (-i)^n e_n,$$

in which e_n is the Hermite function, rescaled as to be of unit length. Check that \wedge is a length preserving map of $\mathbb{L}^2(\mathbb{R})$ onto itself. Give a similar definition of \vee and check that $\hat{f}^\vee = f^{\vee\wedge} = f$.