KTH, Matematik, Maurice Duits

SF2795 Fourier Analysis
Homework Assignment for Lecture 10

1. (Exercise 2.5.7) Prove that $\{h_n\}_{n=0}^{\infty}$ spans $\mathbb{L}_2(\mathbb{R})$. *Hint:* For any $f \in \mathbb{L}_2(\mathbb{R})$, the Fourier transform of $f \exp(-\pi x^2)$ can be expanded in a power series:

$$\begin{aligned}
[f \exp(-\pi x^2)] &= \int f(x) \exp(-\pi x^2) \mathrm{e}^{-2\pi\gamma x} \mathrm{d}x \\
&= \int f(x) \exp(-\pi x^2) \sum_{n=0}^{\infty} (-2\pi \mathrm{i}\gamma)^n (x^n/n!) \mathrm{d}x \\
&= \sum_{n=0}^{\infty} (-2\pi \mathrm{i}\gamma^n/n!) \int f(x) \exp(-\pi x^2) x^n \mathrm{d}x.
\end{aligned}$$

Check that this is legitimate, and infer that $f = 0$ is the only function that is perpendicular to Hermite functions. *Amplification:* To do things this way, you have to know that $[f \exp -\pi x^2]^\wedge$ only if $f = 0$. Prove this by hand. *Hint:*

$$([f \exp -\pi x^2]^\wedge, \hat{g}) = (f \exp(-\pi x^2), g)$$

may be proved by hand for any $g \in C_\downarrow^\infty(\mathbb{R})$.

2. (Exercise: 2.5.8) Define $\hat{f}$ for the general $f \in \mathbb{L}^2(\mathbb{R})$ by Wiener's recipe:

$$\hat{f} = \sum_{n=0}^{\infty} (f, e_n)(-\mathrm{i})^n e_n,$$

in which e_n is the Hermite function, rescaled as to be of unit length. Check that $\wedge$ is a length preserving map of $\mathbb{L}^2(\mathbb{R})$ onto istelf. Give a similar definition of $\vee$ and check that $\hat{f}^\vee = f^{\vee\wedge} = f$.