KS 2 SF1634 VT 16
Svar & Lösningsförslag

(1.) a) $u_n(x,t) = e^{-n^2 t}\sin nx \quad , \quad n \in \mathbb{Z}_+ = \{1,2,3,\dots\}$

$$* \quad \frac{\partial u_n}{\partial x} = -n e^{-n^2 t}\cos nx$$

$$\left.\begin{aligned} \frac{\partial^2 u_n}{\partial x^2} &= -n^2 e^{-n^2 t}\sin nx \\ \frac{\partial u_n}{\partial t} &= -n^2 e^{-n^2 t}\sin nx \end{aligned}\right\} \Rightarrow \underline{\frac{\partial^2 u_n}{\partial x^2} = \frac{\partial u_n}{\partial t}}$$

$$* \quad \underline{u(0,t)} = e^{-n^2 t}\underbrace{\sin(n\cdot 0)}_{=0} = \underline{0}$$

$$* \quad \underline{u(\pi,t)} = e^{-n^2 t}\underbrace{\sin(n\pi)}_{=0} = \underline{0}$$

V.S.B.

b) Bilda $u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$

$$= \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin nx \, ,$$

där $c_n \in \mathbb{R}$ är godtyckliga konstanter.

Enligt superpositionsprincipen uppfyller $u(x,t)$ de homogena ekvationerna (1)-(3).

Sök nu c_n, $n=1,2,\dots$, så att $u(x,t)$ också uppfyller (4)

1. b) (forts.)

$$u(x,0) = \sum_{n=1}^{\infty} c_n e^{-n^2 \cdot 0} \sin nx$$

$$= \sum_{n=1}^{\infty} c_n \sin nx \overset{\text{ska}}{\underset{\text{vara}}{=}} \frac{x}{2} \quad 0 < x < \pi$$

Alltså ska c_n väljas som Fourier-sinus koefficienter till $f(x) = \frac{x}{2}$ på $(0, \pi)$,

$$c_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \cdot \frac{1}{2} \int_0^{\pi} x \sin nx \, dx = \left\{ \begin{matrix} U = x & V = -\frac{1}{n} \cos nx \\ dU = dx & dV = \sin nx \, dx \end{matrix} \right\}$$

$$= \frac{1}{\pi} \left[-\frac{1}{n} x \cos nx \right]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx$$

$$= \frac{1}{\pi} \left[-\frac{1}{n} x \cos nx \right]_0^{\pi} + \frac{1}{n^2} \left[\sin nx \right]_0^{\pi}$$

$$= \frac{1}{n\pi} \left(0 - \pi \cos n\pi \right) + \frac{1}{n^2} \left(\underbrace{\sin n\pi}_{=0} - \underbrace{\sin 0}_{=0} \right)$$

$$= -\frac{1}{n} \cos n\pi = \frac{(-1)^{n+1}}{n}$$

så

$$u(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n^2 t} \sin nx$$

(2.) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega$

Eftersom $\hat{f}(\omega) = H(\omega+1) - H(\omega-1)$

$$= \begin{cases} 0 & |\omega| > 1 \\ 1 & |\omega| < 1 \end{cases}$$

fås

För $t \neq 0$ $f(t) = \frac{1}{2\pi} \int_{-1}^{1} e^{i\omega t} d\omega = \frac{1}{2\pi} \left[\frac{1}{it} e^{i\omega t} \right]_{-1}^{1}$

$$= \frac{1}{2\pi} \frac{1}{t} \cdot \frac{1}{i} \left(e^{it} - e^{-it} \right) =$$

$$= \frac{1}{2\pi t i} 2i \sin t = \frac{\sin t}{\pi t}$$

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \underbrace{e^{i\omega \cdot 0}}_{=1} d\omega = \frac{1}{2\pi} \int_{-1}^{1} 1 \cdot d\omega = \frac{1}{\pi}$$

OBS: $\lim_{t \to 0} f(t) = \lim_{t \to 0} \frac{1}{\pi} \frac{\sin t}{t} = \frac{1}{\pi} \underbrace{\lim_{t \to 0} \frac{\sin t}{t}}_{=1} = \frac{1}{\pi}$

så f är kontinuerlig för alla t,

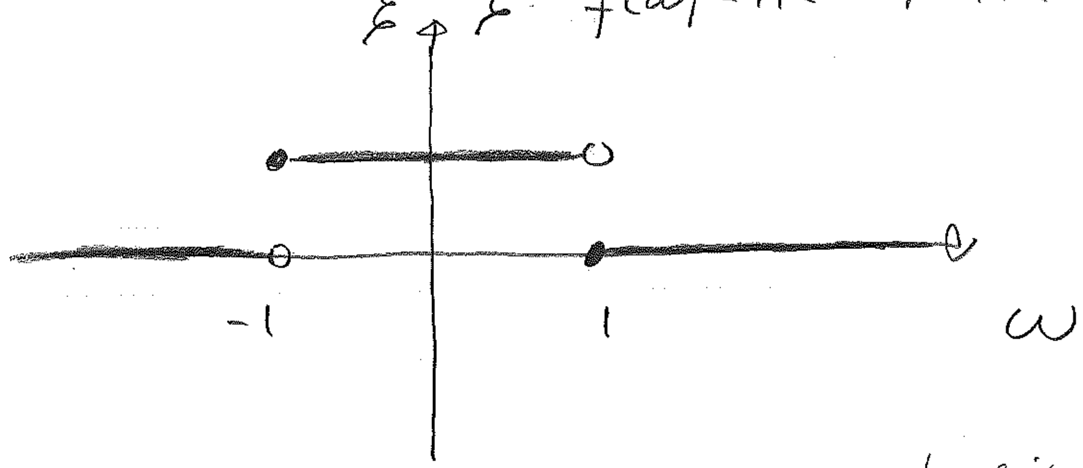
$$f(t) = \begin{cases} \frac{1}{\pi} \frac{\sin t}{t}, & t \neq 0 \\ \frac{1}{\pi} & t = 0 \end{cases}$$

$f(t) = 0$, $\iff \sin t = 0 \iff t = n\pi$

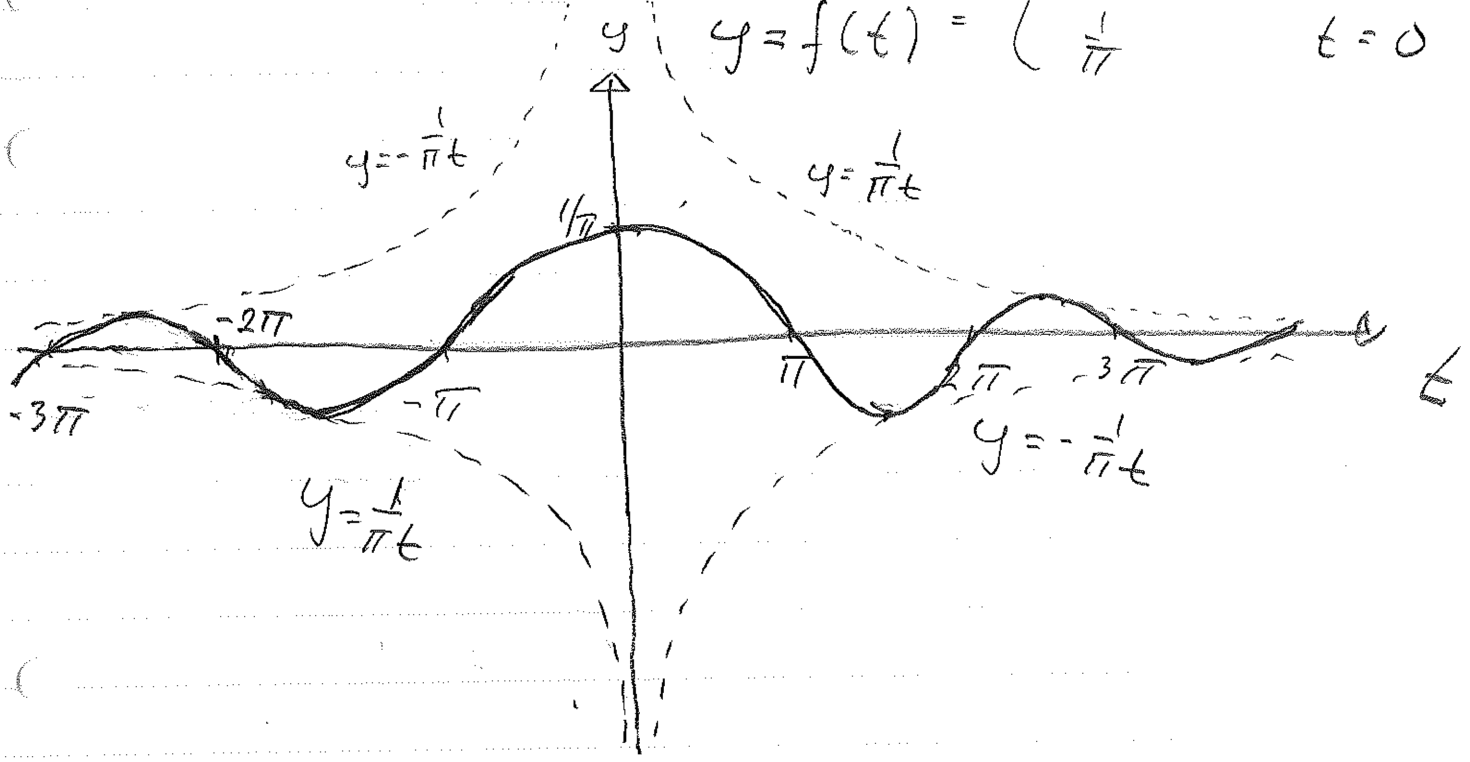
(2.)

(forts.)

$$\xi = \hat{f}(\omega) = H(\omega+1) - H(\omega-1)$$



$$y = f(t) = \begin{cases} \frac{1}{\pi}\,\frac{\sin t}{t} & t \neq 0 \\ \frac{1}{\pi} & t = 0 \end{cases}$$



3. a) $\{f_n(x)\}_{n=1}^{\infty}$ ortogonal på (a,b)

om $\displaystyle\int_a^b f_n(x) f_m(x)dx = 0$

för alla $n \neq m$, $n, m \in \{1,2,3,\dots\}$

b) $\displaystyle\int_0^{\pi} \sin nx \sin mx \, dx$

$$= \int_0^{\pi} \frac{1}{2}\left(\cos(nx-mx) - \cos(nx+mx)\right)dx$$

$$= \frac{1}{2}\int_0^{\pi} \cos(n-m)x \, dx - \frac{1}{2}\int_0^{1} \cos(n+m)x \, dx$$

$$= \{\text{om } n \neq m\} = \frac{1}{2}\left[\frac{1}{n-m}\sin(n-m)x\right]_0^{\pi}$$

$$- \frac{1}{2}\left[\frac{1}{n+m}\sin(n+m)x\right]_0^{\pi} =$$

$$= \frac{1}{2(n-m)}\Big(\underbrace{\sin(n-m)\pi}_{=0} - \underbrace{\sin(n-m)\cdot 0}_{=0}\Big)$$

$$- \frac{1}{2(n+m)}\Big(\underbrace{\sin(n+m)\pi}_{=0} - \underbrace{\sin(n+m)0}_{=0}\Big)$$

(ty $n-m$ och $n+m$ är heltal)

$= 0$

så $\{\sin nx\}_{n=1}^{\infty}$ ortogonal på $(0,\pi)$ V.S.B.