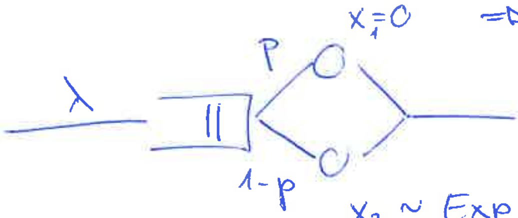
Recitation - Chapter 10 M / G / 1

10.1, 2, 4
Ex. 9

10.1.



$x_1 = 0 \Rightarrow E[x_1] = 0, \; E[x_1^2] = 0, \; V[x_1] = 0$

$x_2 \sim Exp(p) \Rightarrow E[x_2] = \frac{1}{p}, \; E[x_2^2] = \frac{2}{p^2}, \; V[x] = \frac{1}{p^2}$

a) $b(x) = p \cdot 0 + (1-p) \cdot p \cdot e^{-px}$

$E[x] = pE[x_1] + (1-p)E[x_2] = 0 + (1-p) \cdot \frac{1}{p} = \frac{1-p}{p}$

$E[x^2] = pE[x_1^2] + (1-p)E[x_2^2] = \dots \frac{2(1-p)}{p^2}$

$V[x] = E[x^2] - E[x]^2 = \dots = \frac{1-p^2}{p^2}$

b) $W = \frac{\lambda E[x^2]}{2(1-\rho)} = \dots = \frac{\rho}{p(1-\rho)} = \frac{\lambda \frac{2(1-p)}{p^2}}{\cancel{2}\left(1 - \frac{\lambda(1-p)}{p}\right)} = \frac{\rho}{p(1-\rho)} \qquad \rho = \frac{\lambda(1-p)}{p}$

$\underbrace{S^*(s) = B^*(s)}_{\text{different notation for the same thing.}} \triangleq \int_0^\infty e^{-sx} b(x)dx = \int_0^\infty \underbrace{p\,\delta(x)e^{-sx}}_{1} + \int_0^\infty (1-p)\underbrace{pe^{-px}e^{-sx}}_{\exp}dx =$

$= p + (1-p)\frac{p}{p+s} = \dots = \frac{p(1+s)}{p+s} \qquad = \frac{p^2 + ps + p - p^2}{p+s} = \frac{p(1+s)}{p+s}$

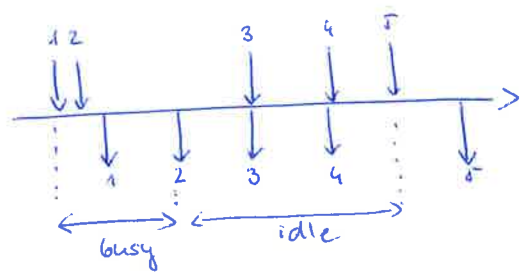
$W^*(s) = \frac{s(1-\rho)}{s - \lambda + \lambda S^*(s)} = \dots = \frac{(1-\rho)(s+p)}{s + p(1+\rho)}$

c) utilization $\triangleq \frac{\lambda_{eff} E[x]}{m} = \lambda E[x] = \rho = \frac{\lambda(1-p)}{p}$

$\lambda_{max}: \; \rho < 1 \Rightarrow \lambda_{max} < \frac{p}{1-p}$

Idle and busy period length

- Only type 2 packets move the system to busy state $\Rightarrow$
- Arrival of type 2: Poisson $(1-p)\lambda$



$\Rightarrow \bar{\tau}_{idle} = \frac{1}{(1-p)\lambda}, \quad \rho = \frac{\bar{\tau}_{busy}}{\bar{\tau}_{busy} + \bar{\tau}_{idle}}, \quad \bar{\tau}_{busy} = \frac{\rho}{1-\rho} \cdot \frac{1}{(1-p)\lambda} = \dots = \frac{1}{p - \lambda(1-p)}$

~~$\bar{\tau}_{busy}$~~

10.2.

$M/M/1 \quad N_q = 8.1$

$M/E_5/1$, same $\rho \rightarrow N_q = ?$

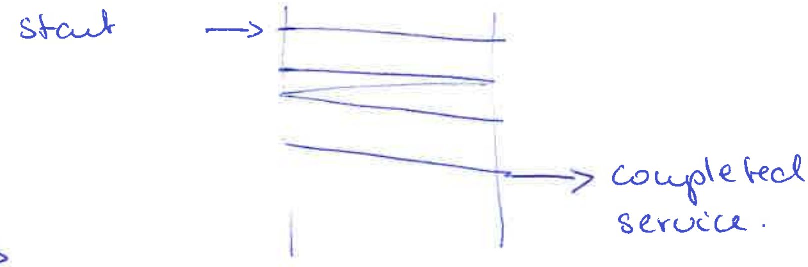
~~=> Need to calculate:~~

a) $M/G/1$: $\bar{N}_q = \lambda W = \dfrac{\lambda \rho E[x]}{2(1-\rho)}(1+C_x^2) = \dfrac{\rho^2}{2(1-\rho)}(1+C_x^2)$

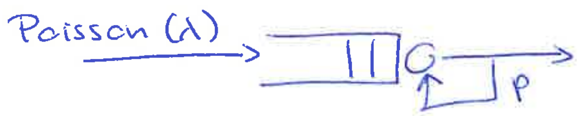
$$\frac{\bar{N}_{q,\,M/E_5/1}}{\bar{N}_{q\,M/M/1}} = \frac{1+\frac{1}{5}}{1+1} = \frac{\frac{6}{5}}{2} = \frac{6}{10} = \frac{3}{5}$$

$\bar{N}_{q\,M/E_5/1} = \frac{3}{5} \cdot \bar{N}_{q\,M/M/1} = 4.86$ (significant decrease!)

b) We skipped the $M/H_2/1$, rather boring.



10.4.



Service? Sequence of tx attempts, each $\tau = 1$

a) Service time distribution?

$P(\text{service time} = n) = P(n \text{ transmissions until success}) =$

$$p^{n-1}(1-p) \Rightarrow N \sim \text{Geom}(1-p)$$

b) Message delay = waiting time + repeated transmission.

System: M/G/1

$$E[x] = \sum_{n=1}^{\infty} n \cdot p^{n-1}(1-p) = (1-p)\sum_{n=1}^{\infty}(p^n)' = (1-p)\left[\sum_{n=1}^{\infty} p^n\right]'$$

$$= (1-p)\left[\frac{p}{1-p}\right]' = (1-p)\,\frac{(1-p)+p}{(1-p)^2} = \frac{1}{1-p}$$

$$E[x^2] = \sum_{n=1}^{\infty} n^2 p^{n-1}(1-p) = \dots = \frac{1+p}{(1-p)^2}$$

[Or: from books: $V[x] = \frac{p}{(1-p)^2}$]

$$T = E[x] + \frac{\lambda E[x^2]}{2(1-\rho)} = \frac{1}{1-p} + \frac{\lambda \frac{1+p}{(1-p)^2}}{2\left(1-\lambda\frac{1}{1-p}\right)} = \dots$$

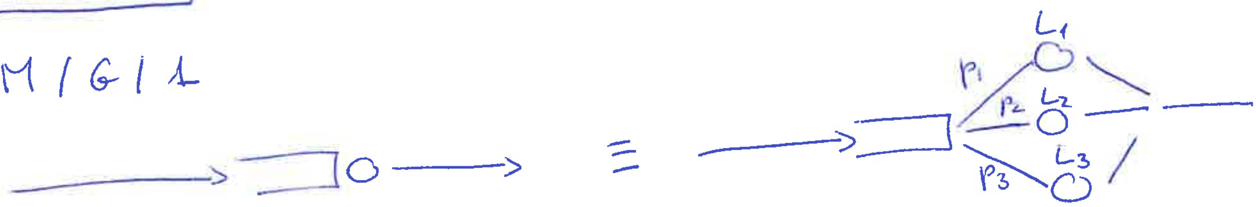
c) λ_{max}: $\rho < 1$

$$\lambda E[x] = \lambda \cdot \frac{1}{1-p} < 1$$

$$\lambda < 1-p$$

$\boxed{\text{Exam 9}}$

M/G/1



$\lambda = 10^4$ packets/s

$C = 34 \cdot 10^6$ b/s

Packet size distribution:

$P(L=40B) = 0.65$
$P(L=595B) = 0.2$
$P(L=1500B) = 0.15$
} deterministic

a) First calculate the respective transmission times:

$\tau_1 = \frac{L_1 \cdot 8}{C} = \dots, \quad \tau_2, \quad \tau_3$ (So we have τ_1, τ_2, τ_3 fixed numbers)

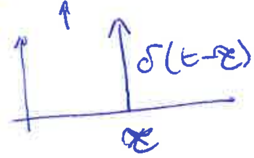
$E[x] = 0.65\,\tau_1 + 0.2\,\tau_2 + 0.15 \cdot \tau_3 \Rightarrow W$

$E[x^2] = 0.65\,\tau_1^2 + 0.2\,\tau_2^2 + 0.15\,\tau_3^2 \Leftarrow$ Why?

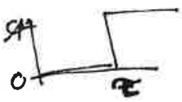
$\Uparrow$ What is the second moment of a deterministic random variable? $P(x=\tau)=1$

② $E[x^2] = \int_0^\infty x^2 f(x)\,dx = \tau^2 \Rightarrow V[x] = E[x^2] - E[x]^2 = 0$ ①

$\delta(t-\tau)$ $\quad f(x) \neq 0$ only at $x = \tau$!



b) What will be $B^*(s)$? $\quad f(x) = \delta(t-\tau)$ $\quad F(x)$:



Laplace transform $\delta(t-\tau) \Longleftrightarrow e^{-\tau s}$ $\quad : B^*(s) = \int_0^\infty f(x) e^{-xs}\,dx = e^{-\tau s}$

$B^*(s) = \sum_{i=1}^{3} p_i\, e^{-\tau_i s}$

What distributions are often involved?

- Exponential
- Geometric
- Deterministic
- Erlang-r
- Hyperexp