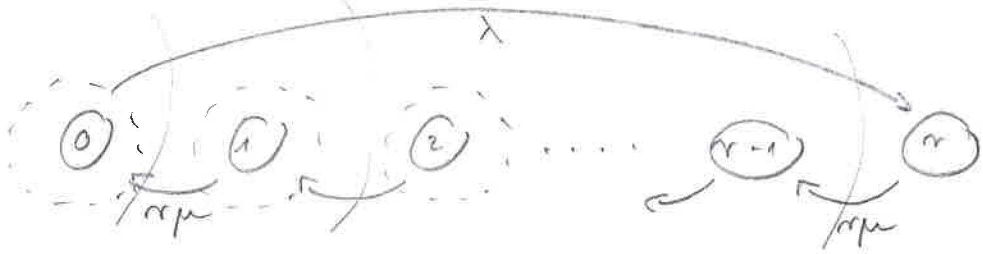
$\boxed{9}$

9.1. $M/E_r/1/1$

$\xrightarrow{\lambda}$ $\boxed{000\ 0}$ $\longrightarrow$ $\qquad \rho = \frac{\lambda}{\mu}$

$p_j = P(j \text{ stages left of service})$ $\qquad \underbrace{}_{x = \frac{1}{\mu}}$ $\quad f_x(t, r, \mu) = \frac{(r\mu t)^{r-1}}{(r-1)!} e^{-r\mu t}$

a)



Both global and local balance equation could work.

1) Global balance equations

$$p_0 \cdot \lambda = p_1 \cdot r\mu \qquad p_1 = \frac{\lambda}{r\mu} p_0$$

$$p_1 \cdot r\mu = p_2 \cdot r\mu \qquad p_{i+1} = p_i \quad i \geq 1$$

$\vdots$

$$p_r \cdot r\mu = p_0 \cdot \lambda \qquad p_r = \frac{\lambda}{r\mu} p_0 \Leftarrow \text{redundant}$$

$$\sum p_i = 1 \Rightarrow \left.\begin{aligned} p_0\left(1 + r \cdot \frac{\lambda}{r\mu}\right) = 1 \\ p_0(1+\rho) = 1 \end{aligned}\right\} \Rightarrow \begin{aligned} p_0 &= \frac{1}{1+\rho} = \frac{\mu}{\mu+\lambda} \\ p_i &= \frac{\rho}{r(1+\rho)} \end{aligned}$$

2) Local balance equations. – cuts

$$\left.\begin{aligned} p_0 \lambda &= p_1 \cdot r\mu \\ p_0 \cdot \lambda &= p_2 r\mu \\ p_0 \lambda &= p_3 \cdot r\mu \\ &\vdots \\ p_0 \lambda &= p_r \cdot r\mu \end{aligned}\right\} p_i = \frac{\lambda}{r\mu} p_0 \ \ldots$$

$$\sum p_i = 1$$

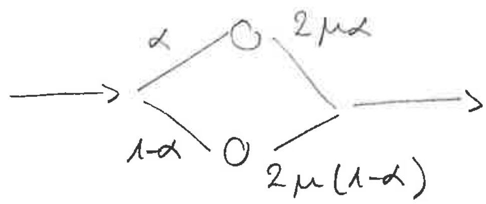
b) What is the probability that the system is busy:

$$P(\text{busy}) = 1 - p_0 = 1 - \frac{1}{1+\rho} = \frac{\rho}{1+\rho} = \text{utilization} = P(\text{blocking})$$

c) Average idle time: $T_{idle} = \frac{1}{\lambda}$
Average busy time: $T_{busy} = \frac{1}{\mu}$

$$\left.\begin{aligned} T_{idle} &= \tfrac{1}{\lambda} \\ T_{busy} &= \tfrac{1}{\mu} \end{aligned}\right\} P(\text{blocking}) = \frac{T_{busy}}{T_{idle} + T_{busy}} = \frac{\frac{1}{\mu}}{\frac{1}{\lambda} + \frac{1}{\mu}} = \frac{\rho}{1+\rho}$$

9.2. $M/H_2/1/1$



a) Service time distribution: mean, second moment, variance

b) Steady state, P(idle), P(busy), P(server 1 used)

a) Service time distribution:

$x_1 \sim Exp(2\mu\alpha) \qquad E[X_1] = \frac{1}{2\mu\alpha}$

$x_2 \sim Exp(2\mu(1-\alpha)) \qquad E[X_2] = \frac{1}{2\mu(1-\alpha)} \Rightarrow$

$$\underline{\underline{E[X]}} = \alpha \cdot E[X_1] + (1-\alpha)E[X_2] = \frac{\alpha}{2\mu\alpha} + \frac{(1-\alpha)}{2\mu(1-\alpha)} = \underline{\underline{\frac{1}{\mu}}}$$

$$E[X^2] = \alpha E[X_1^2] + (1-\alpha) E[X_2^2] = \frac{2\alpha}{4\mu^2\alpha^2} + \frac{2(1-\alpha)}{4\cdot\mu^2(1-\alpha)^2} =$$

$$\frac{1}{2\mu^2\alpha} + \frac{1}{2\mu^2(1-\alpha)} = \frac{1}{2\mu^2\alpha(1-\alpha)}$$

$$V[X] = E[X^2] - E[X]^2 = \frac{1}{2\mu^2\alpha(1-\alpha)} - \frac{1}{\mu^2} = \frac{1-2\alpha(1-\alpha)}{2\mu^2\alpha(1-\alpha)}$$

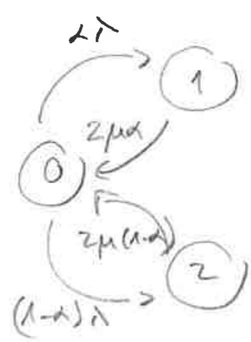
$$C_x = \frac{1-2\alpha(1-\alpha)}{2\alpha(1-\alpha)}$$

$$\left(\text{or} \quad \frac{V[X]}{E[X]^2} = \frac{E[X^2]-E[X]^2}{E[X]^2} = \frac{E[X^2]}{E[X]^2} - 1 = \frac{1}{2\cdot \underbrace{\alpha(1-\alpha)}_{\hat{\ } 0.25}} - 1 \geq 1 \right.$$

0.5

Note: $\alpha = 0.5 \Rightarrow \mu_1 = \mu_2 \Rightarrow$ equivalent exponential server!

b)



$p_0 \cancel{\alpha}\lambda = p_1 2\mu\cancel{\alpha} \qquad p_1 = \frac{\lambda}{2\mu} p_0$

$p_0(1-\alpha)\lambda = p_2 2\mu(\cancel{1-\alpha}) \qquad p_2 = \frac{\lambda}{2\mu} p_0$

$\Rightarrow 1 = p_0\left(1 + \frac{\lambda}{\mu}\right) \Rightarrow p_0 = \frac{1}{1+\rho}$

$p_1 = p_2 = \frac{\rho}{2(1-\rho)}$

$P(\text{idle}) = P(0) = \frac{1}{1+\rho}$

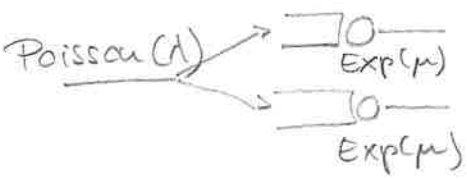
$P(\text{server 1 used}) = p_1$

c) $\bar{N} = ? \qquad - \quad N = 0 \cdot p_0 + 1(p_1 + p_2) = \frac{\rho}{1-\rho}$

Little? $\quad \lambda_{eff} = \lambda \cdot \frac{1}{1+\rho} \qquad \bar{N} = \lambda E[X] = \frac{\lambda}{1+\rho} \cdot \frac{1}{\mu} = \frac{\rho}{1+\rho}$

Exam problems (4)

~ modified: let us look at a)



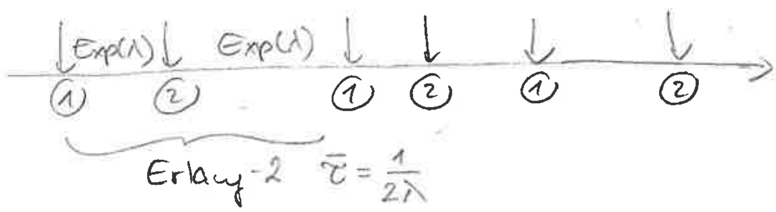
Model the transmission queues

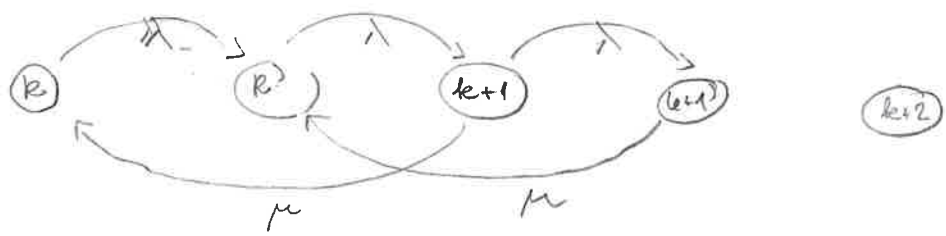
1) Random dispersion $p_1 = p_2 = \frac{1}{2}$

2) Round-robin: 1, 2, 1, 2, ...

1) Random split of Poisson $\Rightarrow$ Poisson$(p_i\lambda)$, M/M/1

2) Round-robin? Inter-arrival times: $E_2 \Rightarrow E_2/M/1$



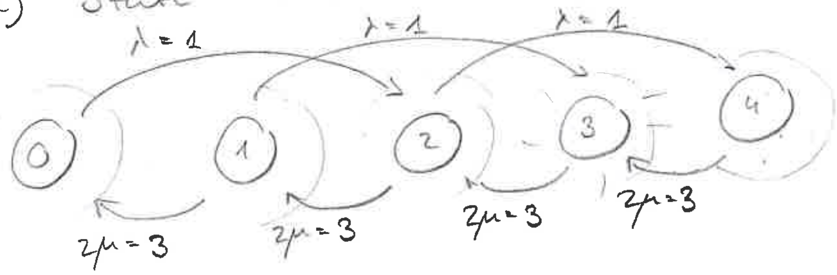


Collection of exam problems - 6

$M/E_2/1/2 \qquad \lambda = 1 \text{ job/minute} \qquad \mu = 1.5 \text{ job/minute}$

a) State transition diagram, state probabilities

global or local balance equations?



$p_0 = 3p_1 \qquad p_1 = \frac{1}{3}p_0$

$4p_1 = 3p_2 \qquad p_2 = \frac{4}{3}p_1 = \frac{4}{9}p_0$

$4p_2 = p_0 + 3p_3 \qquad p_3 = \frac{4p_2 - p_0}{3} = \frac{\frac{16}{9} - \frac{9}{9}}{3}p_0 = \frac{5}{27}p_0$

$p_2 = 3p_4 \qquad p_4 = \frac{p_2}{3} = \frac{4}{27}p_0$

$$1 = p_0\left[1 + \frac{1}{3} + \frac{4}{9} + \frac{5}{27} + \frac{4}{27}\right] = p_0 \frac{27 + 9 + 12 + 5 + 4}{27} = p_0 \frac{57}{27}$$

$$1 = \frac{19}{9}p_0 \Rightarrow \quad p_0 = \frac{9}{19}, \quad p_1 = \frac{3}{19}, \quad p_2 = \frac{4}{19}, \quad p_3 = \frac{5}{19 \cdot 3}, \quad p_4 = \frac{4}{19 \cdot 3}$$

---

b)
$P(\text{blocking}) = p_3 + p_4 = \frac{3}{19}$

- Mean time the system stays in blocking state: ?
  - Blocking state can start in state 3 $\Rightarrow T(\text{block}_3) = \frac{1}{2\mu}$
    state 4 $\Rightarrow T(\text{block}_4) = \frac{1}{2\mu} + \frac{1}{2\mu} = \frac{1}{\mu}$

- What is the probability that the system moves to state 3 or 4? $\Rightarrow$ arrival in state 1 or 2!

$$T(\text{block}) = \frac{p_1}{p_1 + p_2} \cdot \frac{1}{2\mu} + \frac{p_2}{p_1 + p_2} \cdot \frac{1}{\mu} = \frac{3}{7} \cdot \frac{1}{3} + \frac{4}{7} \cdot \frac{2}{3} = \frac{11}{21} =$$

$$T(\text{non-block}) \Rightarrow \quad P(\text{blocking}) = \frac{T(\text{blocking})}{T(\text{blocking}) + T(\text{non-blocking})} = \dots$$

$$\frac{3}{19} = \frac{\frac{11}{21}}{\frac{11}{21} + T(\text{non-blocking})} = \dots$$

c) Is OK in the compendium!