M / M / m - Erlang wait system

1. System def., block diagram, Markov chain
2. Steady state p, stability?
3. Performance

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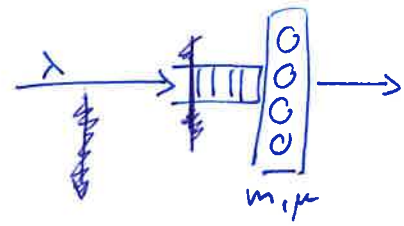
① - Arrival : Poisson (λ)

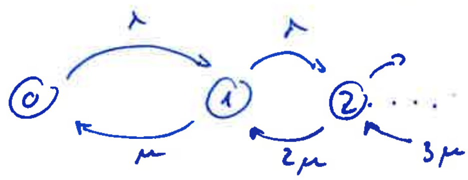
- Service : Exp (μ) $\quad x = \frac{1}{\mu}$
- Servers : m
- Buffer : ∞

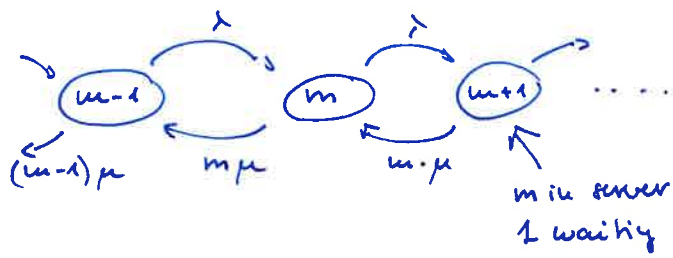


Offered load : $a = \lambda \cdot \bar{x} = \frac{\lambda}{\mu}$

Server utilization = $\frac{a}{m}$

- Markov - chain representation : state : number of customers





② Steady state p - local balance equations

$k \leq m \qquad \left.\begin{aligned} p_0 \cdot \lambda &= p_1 \mu \\ p_1 \cdot \lambda &= p_2 \cdot 2\mu \\ &\vdots \\ p_{m-1}\lambda &= p_m \cdot m \cdot \mu \end{aligned}\right\}$ like M/M/m/m $\quad p_k = \frac{a^k}{k!} p_0$

$k > m \qquad \left.\begin{aligned} p_m \lambda &= p_{m+1} m\mu \\ p_{m+1}\lambda &= p_{m+2} m \cdot \mu \\ &\vdots \end{aligned}\right\} \begin{aligned} p_{m+1} &= p_m \cdot \frac{a}{m} = \frac{a}{m} \cdot \frac{a^m}{m!} p_0 \\ p_{m+2} &= \frac{a}{m} p_{m+1} = \left(\frac{a}{m}\right)^2 \cdot \frac{a^m}{m!} p_0 \end{aligned} \quad \Rightarrow \quad p_k = \frac{a^k}{m^{k-m} m!} p_0$

$$\sum_{i=0}^{\infty} p_i = 1 \Rightarrow p_0 \left[\sum_{i=0}^{m-1} \frac{a^i}{i!} + \sum_{i=m}^{\infty} \frac{a^i}{m^{i-m} \cdot m!} \right] = \dots = p_0 \left[\sum_{i=0}^{m-1} \frac{a^i}{i!} + \frac{a^m/m!}{1 - a/m} \right] = 1$$

$\Rightarrow p_0 \Rightarrow p_k$

$\boxed{\begin{aligned} &\text{Stability:} \\ &a/m < 1 \\ &a < m \end{aligned}}$

3. Performance

Probability ~~of waiting~~ that the arriving customer has to wait

P_{wait} = P(customer arrives when system is in state m, m+1 ....) $= \sum_{k=m}^{\infty} p_k$

Derive at home!

$$= \dots\dots = \frac{\frac{a^m/m!}{1 - a/m}}{\sum_{i=0}^{m-1} \frac{a^i}{i!} + \frac{a^m/m!}{1-a/m}} = D_m(a)$$

$C(a,m)$

$\boxed{\text{Erlang-C form}}$

No closed form. Can we use Erlang tables?

$$D_m(a) = \frac{m E_m(a)}{m - a(1 - E_m(a))}$$

, $E_m(a)$ from tables. See recitation problems.

E.g.
$m=10$, $a=4$: $E_m(a) = 0.005$

$$D_m(a) = \frac{10 \cdot 0.005}{10 - 4 \cdot 0.995} = \cdot = 0.0083$$

Average performance measures

$N_s = \lambda \bar{x} = a$

$$N_q = \sum_{k=m+1}^{\infty} (k-m) p_k = \dots = D_m(a) \frac{a}{m-a}$$

(formula sheet: $a = \rho$)

$N := N_s + N_q$

Waiting time distribution

Recall M/M/1

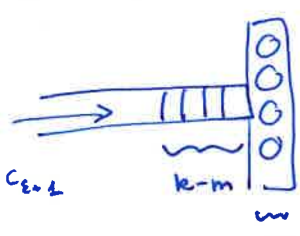
- waiting time for state k in $\mathcal{L}$ domain
- unconditional waiting time
- inverse laplace

~~Waiting states 1, 2~~ ...

$$\mathcal{L}(f_w(t)) = \sum_{k=0}^{\infty} \left(\frac{\mu}{s+\mu}\right)^k \cdot p_k$$

$$W(t) = 1 - \rho \cdot e^{-(\mu-\lambda)\cdot t}$$

M/M/m



$k < m$: no waiting

$k \geq m$:

- number of services that needs to be completed:
  $k - (m-1)$ ← this can be still under service
- time between ~~service~~ end-of-services:
  m parallel service
  $Exp(m\mu) \longrightarrow \mathcal{L}(f_e(t)) = \frac{m\mu}{s+m\mu}$

$$\Rightarrow \mathcal{L}(f_w(t)) - \sum_{k=0}^{\infty} \mathcal{L}(f_w(t|k)) \cdot p_k = \cancel{\sum_{k=m}^{\infty} \left(\frac{m\mu}{s+m\mu}\right)^{k-(m-1)} p_k} \Rightarrow W(t) = 1 - D_m(a) e^{-(m\mu-\lambda)t}$$

$\mathcal{L}(f_w(t|k)) = \left(\frac{m\mu}{s+m\mu}\right)^{k-(m-1)}$ $k \geq m$ $\quad \mathcal{L}(f_w(t|k)) = 1$ $k < m$

$F(s) = \int_0^{\infty} f(t) e^{-st} dt$

$\int_0^{\infty} \delta(t) e^{-st} = 1$

Engset loss systems

Summary

M/M/1
M/M/1/K
M/M/m/m (Erlang loss system)
M/M/m (Erlang wait system)

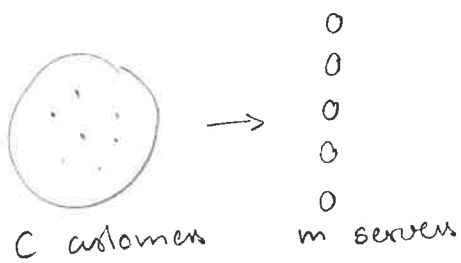
$\lambda_i = \lambda$
Poisson arrival
~ independent requests from a large population.

What happens if the population is small?

$\Rightarrow$ M/M/m/m/C Engset loss system

(extensions on the notation, e.g. M/M/m/K/C)

1 System definition



$m < C \ll \infty$

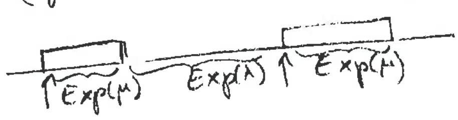
→ Customer does not generate new request while a request is under service

→ The "intensity" of request arrival depends on the number of requests under service.

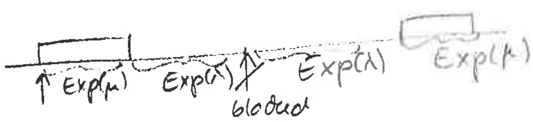
Modeling finite population

Single customer
- thinking (idle) time: Exp (λ)
- request service time: Exp (μ)

(γ in the Virtamo notes)

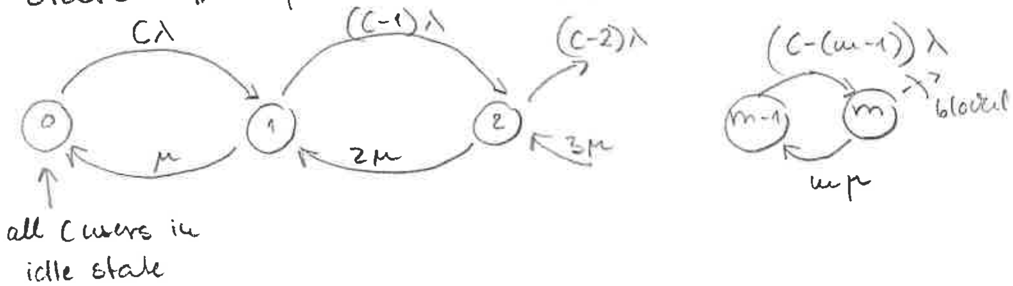


Blocked request
- does not try again
- starts a new thinking time



(2) System model: Markov chain

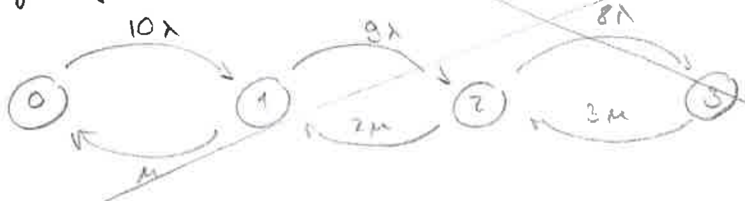
State: # requests under service (or waiting in the M/M/m/K/c case)



$\lambda_i = (c-i)\lambda$

$\mu_i = i\mu$

Group work: ~~M/M/3/3/10, λ, μ~~



3 ~~[illegible]~~

$\lambda = 3 \Leftarrow$ av. thinking time: 20 min

$\mu = 6 \Leftarrow$ av. time on the phone: 10 min

thinking time: Exp(λ)

service time Exp(μ)

$\bar{\tau} = 20$ min

$\bar{x} = 10$ min

(3) State probabilities in steady state

p_k: an independent observer finds the system in state k

$$\left.\begin{aligned} c\lambda p_0 &= \mu p_1 \\ (c-1)\lambda p_1 &= 2\mu p_2 \\ (c-2)\lambda p_2 &= 3\mu p_3 \\ &\vdots \end{aligned}\right\}$$

$$p_1 = \frac{c\lambda}{\mu} p_0$$

$$p_2 = \frac{(c-1)\lambda}{2\mu} p_1 = \frac{c(c-1)\lambda^2}{2\mu^2} p_0$$

$$p_3 = \frac{(c-2)\lambda}{3\mu} p_2 = \frac{c(c-1)(c-2)}{3!}\left(\frac{\lambda}{\mu}\right)^3 p_0 = \frac{c!}{(c-3)!\,3!}\left(\frac{\lambda}{\mu}\right)^3 p_0$$

$$\sum_{k=0}^{m} p_k = 1 \qquad \Rightarrow \qquad p_k = \binom{c}{k}\left(\frac{\lambda}{\mu}\right)^k p_0$$

$$p_0 \sum_{k=0}^{m} \binom{c}{k}\left(\frac{\lambda}{\mu}\right)^k = 1$$

$$p_k = \frac{\binom{c}{k}\left(\frac{\lambda}{\mu}\right)^k}{\sum_{i=0}^{m}\binom{c}{i}\left(\frac{\lambda}{\mu}\right)^i} \quad : \quad \text{Engset distribution}$$

a_k: an arriving customer finds the system in state k: PASTA does not hold! $a_2 \neq p_2$!

$$a_k = P(\text{state } k \text{ at time } t \mid \text{arrival } (t, t+\Delta t)) = \frac{P(\text{state } k, \text{arrival})}{P(\text{arrival})} = \frac{P(\text{arrival} \mid \text{state } k)\,P(\text{state } k)}{\sum_{i=0}^{m} P(\text{arrival} \mid \text{state } i)\,P(\text{state } i)}$$

$$= \frac{\lambda_k \Delta t\, p_k}{\sum_{i=0}^{m} \lambda_i \Delta t\, p_i} = \frac{\lambda_k p_k}{\sum_{i=0}^{m} \lambda_i p_i} = \boxed{\frac{(c-k)p_k}{\sum_{i=0}^{m}(c-i)p_i}}$$

(4) Blocking probability

New definition:

Time blocking = {portion of time the system spends in blocking state} = P_m

Call blocking = P(arriving request gets blocked) = $a_m = \dfrac{\lambda_m p_m}{\sum_{i=0}^{m} \lambda_i p_i}$

Simple form of a_m

$$\boxed{a_m(C) = \frac{(C-m)\lambda \dfrac{\binom{C}{m}\left(\frac{\lambda}{\mu}\right)^m}{\sum_{j=0}^{m}\binom{C}{j}\left(\frac{\lambda}{\mu}\right)^j}}{\sum_{i=0}^{m}(C-i)\lambda \dfrac{\binom{C}{i}\left(\frac{\lambda}{\mu}\right)^i}{\sum_{j=0}^{m}\binom{C}{j}\left(\frac{\lambda}{\mu}\right)^j}} = \frac{(C-m)\binom{C}{m}\left(\frac{\lambda}{\mu}\right)^m}{\sum_{i=0}^{m}(C-i)\binom{C}{i}\left(\frac{\lambda}{\mu}\right)^i}}$$

$$(C-i)\binom{C}{i} = \cancel{(C-i)}\frac{C!}{\cancel{(C-i)!}\,i!} = \frac{C!}{(C-i-1)!\,i!} = \frac{C\,(C-1)!}{(C-1-i)!\,i!} = C\cdot\binom{C-1}{i}$$

$$= \frac{\binom{C-1}{m}\left(\frac{\lambda}{\mu}\right)^m}{\sum_{i=0}^{m}\binom{C-1}{i}\left(\frac{\lambda}{\mu}\right)^i} = p_m(C-1)$$

[Happens to be ...]

(5) Performance measures

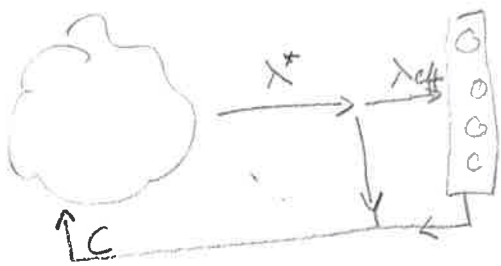
Offered traffic: $\lambda^* = \sum_{i=0}^{m} \lambda_i p_i = \sum_{i=0}^{m} (C-i)\lambda p_i$

Effective traffic: $\lambda_{eff} = \sum_{i=0}^{m-1} \lambda_i p_i$

$\bar{x} = \frac{1}{\mu}$

$N = N_S = \lambda_{eff} \cdot \bar{x}$

$T = x$



(6) When should we assume finite population?

~~if $p_m \approx a_m \to \binom{C}{m} \approx \binom{C-1}{m}$~~

Rule of thumb: $C > 10m \Rightarrow$ infinite assumption with $\lambda^* = C \cdot \lambda$

A simple example

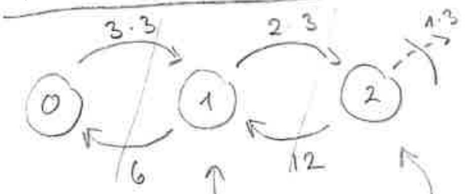
- 3 secretaries
- 2 phones
- each secretary: - tries to make a phonecall after in average 20 minutes paperwork. $\Rightarrow \lambda = 3$ attep/hour
  - is on the phone for 10 minutes in average $\Rightarrow \mu = 6$ calls/hour

System $M/M/2/2/3$

$\lambda = 3$

$\mu = 6$

Markov chain



one secretary on the phone

2 secret. on the phone

$p_0 \cdot 9 = p_1 \cdot 6 \qquad p_1 = \frac{9}{6} p_0 = \frac{3}{2} p_0$

$p_1 \cdot 6 = p_2 \cdot 12 \qquad p_2 = \frac{1}{2} p_1 = \frac{3}{4} p_0$

$p_0 + p_1 + p_2 = 1$

$p_0 \left(1 + \frac{3}{2} + \frac{3}{4} \right) = 1$

$p_0 \left(\frac{4+6+3}{4} \right) = 1 \Rightarrow \{p_0, p_1, p_2\} = \left\{ \frac{4}{13}, \frac{6}{13}, \frac{3}{13} \right\}$

$\Rightarrow$ Both of the phones is occupied in $\frac{3}{13}$ fraction of time

- Probability that a call attempt is blocked? $\Rightarrow a_m$

$$a_m = \frac{\lambda \cdot p_2}{3 \cdot p_0 + 2 \cdot p_1 + 1 \cdot p_2} = \frac{3/13}{12/13 + 12/13 + 3/13} = \frac{3}{27} \ll \frac{3}{13}$$

- How many calls are performed in average in 8 hours?

$$\lambda_{eff} = \sum_{i=0}^{m-1} \lambda_i p_i = 3 \cdot 3 \cdot p_0 + 2 \cdot 3 \cdot p_1 = \frac{36}{13} + \frac{36}{13} = \frac{72}{13}$$

\# calls in 8 hours: $\lambda_{eff} \cdot 8 = \frac{72 \cdot 8}{13}$