Exercise 7, section 1.4 Given any $f \in L^2(\mathbb{S}^1)$ we wish to show that

$$\lim_{n\uparrow\infty} \frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right) = \widehat{f}(0) = \int_0^1 f$$

in the sense of L^2-convergence. Let $g_n(x) = (1/n)\sum_{k=0}^{n-1} f(x+k/n)$. Since the $\wedge$-map is linear, i.e. $\widehat{cg_1+g_2} = c\widehat{g_1}+\widehat{g_2}$ for constant c, we have that

$$\widehat{g}_n(m) = \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1 f\left(x+\frac{k}{n}\right)e^{-2\pi imx}dx.$$

Doing a change of variables $y = x+(k/n)$ and using the fact that our function is 1-periodic we get

$$\widehat{g}_n(m) = \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1 f(y)e^{-2\pi im(y-(k/n))}dy = \frac{\widehat{f}(m)}{n}\sum_{k=0}^{n-1} e^{2\pi im(k/n)}.$$

We note the following; if $m = jn$ for some integer j then $e^{2\pi im(k/n)} = e^{2\pi ijk} = 1$ and thus $\widehat{g}_n(jn) = \widehat{f}(jn)$. If $m \neq jn$ for any integer j then the sum evaluates to zero; indeed we have that

$$\sum_{k=0}^{n-1} e^{2\pi im(k/n)} = \frac{e^{2\pi im}-1}{e^{2\pi im/n}-1} = 0$$

Using Plancheral's identity, we have that

$$\begin{aligned}\|g_n - \widehat{f}(0)\|_2^2 &= \|\widehat{g}_n - \widehat{\widehat{f}(0)}\|^2 \\ &= |\widehat{g}_n(0)-\widehat{f}(0)|^2 + \sum_{m\neq 0}|\widehat{g}_n(m)|^2 \\ &= \sum_{j\neq 0}|\widehat{f}(jn)|^2 \\ &\leq \sum_{|m|\geq n}|\widehat{f}(m)|^2 \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

The limit is true since $f \in L^2(\mathbb{S}^1)$ and hence $\sum_{m\in\mathbb{Z}}|\widehat{f}(m)|^2 < \infty$.