DD2448 Foundations of Cryptography
Lecture 3

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Linear Cryptanalysis of the SPN

Basic Idea – Linearize

Find an expression of the following form with a high probability of occurrence.

$$P_{i_1} \oplus \cdots \oplus P_{i_p} \oplus C_{j_1} \oplus \cdots \oplus C_{j_c} = K_{\ell_1,s_1} \oplus \cdots \oplus K_{\ell_k,s_k}$$

Each random plaintext/ciphertext pair gives an estimate of

$$K_{\ell_1,s_1} \oplus \cdots \oplus K_{\ell_k,s_k}$$

Collect many pairs and make a better estimate based on the majority vote.

How do we come up with the desired expression?

How do we compute the required number of samples?

Bias

Definition. The bias $\epsilon(X)$ of a binary random variable X is defined by

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$\approx 1/\epsilon^2(X)$ samples are required to estimate X (Matsui)

Linear Approximation of S-Box (1/3)

Let X and Y be the input and output of an S-box, i.e.

$$Y = S(X) \ .$$

We consider the bias of linear combinations of the form

$$a \cdot X \oplus b \cdot Y = \left(\bigoplus_i a_i X_i \right) \oplus \left(\bigoplus_i b_i Y_i \right) \ .$$

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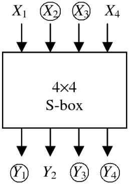
Example: $X_2 \oplus X_3 = Y_1 \oplus Y_3 \oplus Y_4$
The expression holds in 12 out of the 16 cases. Hence, it has a bias of $(12 - 8)/16 = 4/16 = 1/4$.



Linear Approximation of S-Box (2/3)

- Let $N_L(a, b)$ be the number of zero-outcomes of $a \cdot X \oplus b \cdot Y$.
- The bias is then

$$\epsilon(a \cdot X \oplus b \cdot Y) = \frac{N_L(a, b) - 8}{16} \ ,$$

since there are four bits in X, and Y is determined by X.

Linear Approximation Table (3/3)

$$N_L(a,b) - 8$$

Input Sum \ Output Sum	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	+8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	−2	−2	0	0	−2	+6	+2	+2	0	0	+2	+2	0	0
2	0	0	−2	−2	0	0	−2	−2	0	0	+2	+2	0	0	−6	+2
3	0	0	0	0	0	0	0	0	+2	−6	−2	−2	+2	+2	−2	−2
4	0	+2	0	−2	−2	−4	−2	0	0	−2	0	+2	+2	−4	+2	0
5	0	−2	−2	0	−2	0	+4	+2	−2	0	−4	+2	0	−2	−2	0
6	0	+2	−2	+4	+2	0	0	+2	0	−2	+2	+4	−2	0	0	−2
7	0	−2	0	+2	+2	−4	+2	0	−2	0	+2	0	+4	+2	0	+2
8	0	0	0	0	0	0	0	0	−2	+2	+2	−2	+2	−2	−2	−6
9	0	0	−2	−2	0	0	−2	−2	−4	0	−2	+2	0	+4	+2	−2
A	0	+4	−2	+2	−4	0	+2	−2	+2	+2	0	0	+2	+2	0	0
B	0	+4	0	−4	+4	0	+4	0	0	0	0	0	0	0	0	0
C	0	−2	+4	−2	−2	0	+2	0	+2	0	+2	+4	0	+2	0	−2
D	0	+2	+2	0	−2	+4	0	+2	−4	−2	+2	0	+2	0	0	+2
E	0	+2	+2	0	−2	−4	0	+2	−2	0	0	−2	−4	+2	−2	0
F	0	−2	−4	−2	−2	0	+2	0	0	−2	+4	−2	−2	0	+2	0

This gives linear approximation for one round.

How do we come up with linear approximation for more rounds?

Piling-Up Lemma

Lemma. Let $X_1, \ldots, X_t$ be independent binary random variables and let $\epsilon_i = \epsilon(X_i)$. Then

$$\epsilon\left(\bigoplus_i X_i\right) = 2^{t-1}\prod_i \epsilon_i \enspace .$$

Proof. Case $t = 2$:

$$\begin{aligned}
\Pr[X_1 \oplus X_2 = 0] &= \Pr[(X_1 = 0 \wedge X_1 = 0) \vee (X_1 = 1 \wedge X_1 = 1)] \\
&= (\frac{1}{2} + \epsilon_1)(\frac{1}{2} + \epsilon_2) + (\frac{1}{2} - \epsilon_1)(\frac{1}{2} - \epsilon_2) \\
&= \frac{1}{2} + 2\epsilon_1\epsilon_2 \enspace .
\end{aligned}$$

By induction $\Pr[X_1 \oplus \cdots \oplus X_t = 0] = \frac{1}{2} + 2^{t-1}\prod_i \epsilon_i$

Linear Trail

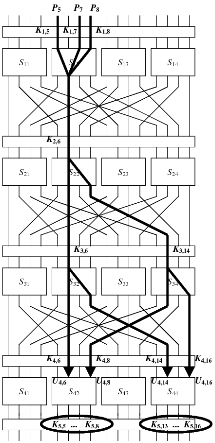
Four linear approximations with $|\epsilon_i| = 1/4$

$$\begin{array}{ll} S_{12}: & X_1 \oplus X_3 \oplus X_4 = Y_2 \\ S_{22}: & X_2 = Y_2 \oplus Y_4 \\ S_{32}: & X_2 = Y_2 \oplus Y_4 \\ S_{34}: & X_2 = Y_2 \oplus Y_4 \end{array}$$

Combine them to get:

$$U_{4,6} \oplus U_{4,8} \oplus U_{4,14} \oplus U_{4,16} \oplus P_5 \oplus P_7 \oplus P_8 = \bigoplus K_{i,j}$$

with bias $|\epsilon| = 2^{4-1}(\frac{1}{4})^4 = 2^{-5}$



Attack Idea

- Our expression (with bias 2^{-5}) links plaintext bits to input bits to the 4th round
- Partially undo the last round by guessing the last key. Only 2 S-Boxes are involved, i.e., $2^8 = 256$ guesses
- For a correct guess, the equation holds with bias 2^{-5}. For a wrong guess, it holds with bias zero (i.e., probability close to $1/2$).

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Required pairs $2^{10} \approx 1000$
Attack complexity $2^{18} \ll 2^{32}$ operations

Linear Cryptanalysis Summary

1. Find linear approximation of S-Boxes.
2. Compute bias of each approximation.
3. Find linear trails.
4. Compute bias of linear trails.
5. Compute data and time complexity.
6. Estimate key bits from many plaintext-ciphertexts pairs.

Linear cryptanalysis is a **known plaintext attack**.

Ideal Block Cipher

Negligible Functions

Definition. A function $\epsilon(n)$ is negligible if for every constant $c > 0$, there exists a constant n_0, such that

$$\epsilon(n) < \frac{1}{n^c}$$

for all $n \geq n_0$.

Motivation. Events happening with negligible probability can not be exploited by polynomial time algorithms! (they "never" happen)

Pseudo-Random Function

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Definition. A family of functions $F : \{0,1\}^k \times \{0,1\}^n \to \{0,1\}^n$ is pseudo-random if for all polynomial time oracle adversaries A

$$\left| \Pr_K \left[A^{F_K(\cdot)} = 1 \right] - \Pr_{R:\{0,1\}^n \to \{0,1\}^n} \left[A^{R(\cdot)} = 1 \right] \right|$$

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$$\left| \Pr_K \left[A^{P_K(\cdot), P_K^{-1}(\cdot)} = 1 \right] - \Pr_{\Pi \in \mathcal{S}_{2^n}} \left[A^{\Pi(\cdot), \Pi^{-1}(\cdot)} = 1 \right] \right|$$

is negligible, where $\mathcal{S}_{2^n}$ is the set of permutations of $\{0,1\}^n$.

Idealized Four-Round Feistel Network

Definition. Feistel round (H for "Horst Feistel").

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Theorem. (Luby and Rackoff) If F is a pseudo-random family of functions, then

$$H_{F_{k_1},F_{k_2},F_{k_3},F_{k_4}}(x) = H_{F_{k_4}}(H_{F_{k_3}}(H_{F_{k_2}}(H_{F_{k_1}}(x))))$$

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Why do we need four rounds?