Recitation 5 M/M/1 (3.3, 5.1, 5.2, 5.6)

5.1. M/M/1

Arrival: Poisson(λ)

Service: Exp($C \cdot \mu$) & $\bar{x} = \frac{L}{C} = \frac{1}{\mu C}$ $\mu' = \mu C$

$\boxed{\text{Note! } \mu C > \lambda \text{ for stability!}}$

a) $C = ?$ $\frac{1}{\mu' - \lambda} < T_0$

$$\frac{1}{\mu C - \lambda} < T_0$$

$$\frac{1}{T_0} < \mu C - \lambda$$

$$\frac{\frac{1}{T_0} + \lambda}{\mu} < C$$

$$P(T > 3T_0) = e^{-(\mu' - \lambda) \cdot 3T_0} = e^{-\frac{1}{T_0} 3T_0} = e^{-3} \approx 0.05$$

b) $C = ?$

$$P(T > t) < p$$

$$e^{-(\mu C - \lambda)t} < p$$

$$-(\mu C - \lambda)t < \ln(p)$$

$$C > \left(\frac{-\ln(p)}{t} + \lambda\right) / \mu$$

c) We have to use the same equation, but express λ

$$\lambda < \frac{\ln(p)}{t} + \mu C$$

$p_k = (1-\rho)\rho^k$

$N = \frac{\rho}{1-\rho}$

$T = \frac{1}{\mu - \lambda}$

$T \sim Exp(\mu - \lambda)$

$F_T(t) = 1 - e^{-(\mu-\lambda)t}$

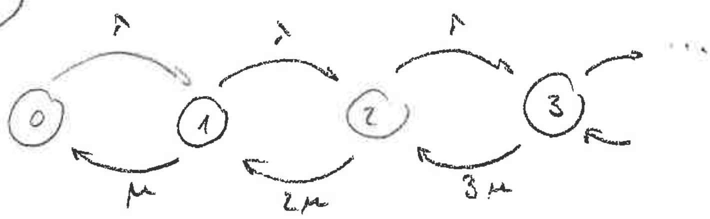
Little: $N = \lambda T$

Start with: 3.3.

Do not do: 5.4, 5.5

M/M/1

5.6



Def: $\rho = \frac{\lambda}{\mu}$

$p_0 \lambda = p_1 \mu$

$p_1 \lambda = 2\mu p_2$

$\lambda p_2 = 3\mu p_3$

$\vdots$

$p_1 = \frac{\lambda}{\mu} p_0$

$p_2 = \frac{\lambda^2}{2\mu^2} p_1$

$p_3 = \frac{\lambda^3}{2 \cdot 3 \cdot \mu^3} p_2$

$\vdots$

$p_k = \frac{\lambda^k}{k!\,\mu^k} p_0$

$\sum_{k=0}^{\infty} p_k = 1$

$$p_0\left(1 + \rho + \frac{\rho^2}{2} + \frac{\rho^3}{3!} + \frac{\rho^4}{4!} \dots\right) = 1$$

$$p_0 \underbrace{\sum_{k=0}^{\infty} \frac{\rho^k}{k!}}_{e^{\rho}} = 1 \Rightarrow p_0 = e^{-\rho} \Leftarrow$$ Note: not $1-\rho$!

$$p_k = \frac{\rho^k}{k!} e^{-\rho}$$

Poisson distribution!

$$\underline{\underline{N}} = \sum_{k=0}^{\infty} k \cdot p_k = \sum_{k=1}^{\infty} k \cdot \frac{\rho^k}{k!} e^{-\rho} = e^{-\rho} \cdot \rho \sum_{k=1}^{\infty} \frac{\rho^{k-1}}{(k-1)!} = e^{-\rho} \cdot \rho \cdot e^{\rho} = \underline{\underline{\rho}}$$

$$\underline{\underline{T}} = \frac{N}{\lambda} = \frac{\lambda}{\mu} \cdot \frac{1}{\lambda} = \underline{\underline{\frac{1}{\mu}}}$$

Note: $\bar{x} \neq \frac{1}{\mu} \Rightarrow$ we have to calculate from the MC!

$$\underline{\underline{N_q}} = \underbrace{\sum_{k=1}^{\infty} (k-1) \frac{\rho^k}{k!} e^{-\rho}}_{\text{Divide into two sums}} = \sum_{k=1}^{\infty} k \frac{\rho^k}{k!} e^{-\rho} - \underbrace{\sum_{k=1}^{\infty} \frac{\rho^k}{k!} e^{-\rho}}_{1 - e^{-\rho}} = \rho + e^{-\rho} - 1$$

$$\underline{\underline{W}} = \frac{N_q}{\lambda} =$$

$$\underline{\underline{N_s}} = 1 - e^{-\rho}$$

$$\underline{\underline{\bar{x}}} = \frac{1 - e^{-\rho}}{\lambda}$$

Offered load: $a = \lambda \bar{x} = 1 - e^{-\rho}$

5.2.

c) $T = \frac{N}{\lambda} = \frac{\sum_{k=0}^{\infty} k p_k}{\lambda}$

a) $T = \sum_{k=0}^{\infty} E[T|k] p_k = \sum_{k=0}^{\infty} \frac{k+1}{\mu} p_k = \frac{1}{\mu}\left(\sum_{k=0}^{\infty} k \cdot p_k + \sum_{k=0}^{\infty} p_k\right) = \frac{1}{\mu}(\bar{N}+1) = \dots = \underline{\underline{\frac{1}{\mu-\lambda}}}$

b) Last Come First Served

$T =$ P(empty system) $T_{\text{own service}}$ +

P(non empty system) $\Big[\underbrace{T_{\text{ongoing service}}}_{\tilde{T}_1} + \overbrace{T_{\text{service of jobs arriving under ongoing service after me}}}^{\tilde{T}_2}$

$+ T_{\text{service of those arriving while serving these}} + \dots + T_{\text{own service}} \Big] =$

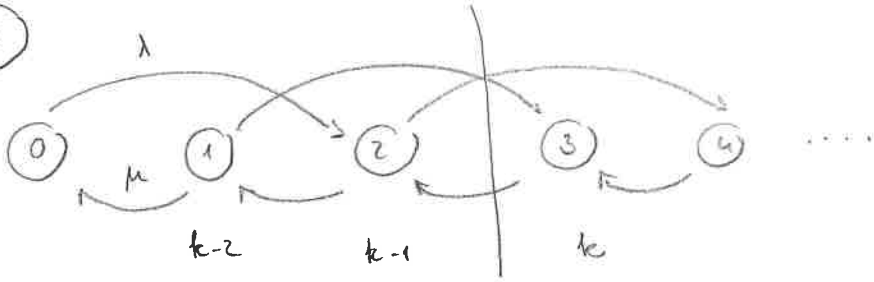
$$p_0 \cdot \bar{x} + (1-p_0)\left(\underbrace{\bar{x} + (\lambda\bar{x})\bar{x} + \lambda(\lambda\bar{x}\bar{x})\bar{x} + \dots}_{} + \bar{x}\right) =$$

$$p_0 \bar{x} + (1-p_0)\left(\sum_{i=0}^{\infty} (\lambda x)^i \bar{x} + \bar{x}\right) =$$

$$p_0 \bar{x} + (1-p_0)\left[\frac{1}{1-\lambda\bar{x}}\bar{x} + \bar{x}\right] = \left(1-\frac{\lambda}{\mu}\right)\cdot\frac{1}{\mu} + \left(\frac{\lambda}{\mu}\right)\left[\frac{1}{1-\frac{\lambda}{\mu}}\cdot\frac{1}{\mu} + \frac{1}{\mu}\right] =$$

$$\frac{1}{\mu} - \cancel{\frac{\lambda}{\mu^2}} + \frac{\lambda}{\mu}\frac{1}{\mu-\lambda} + \cancel{\frac{\lambda}{\mu^2}} = \frac{\mu-\lambda+\lambda}{\mu(\mu-\lambda)} = \underline{\underline{\frac{1}{\mu-\lambda}}}$$

5.5



$$\lambda p_0 = \mu p_1$$

$$\lambda p_{k-2} + \lambda p_{k-1} = \mu p_k \qquad / \cdot z^k, \text{ add up from } k=2$$

$$\sum_{k=2}^{\infty} z^k (\lambda p_{k-2} + \lambda p_{k-1}) = \sum_{k=2}^{\infty} z^k \mu p_k \qquad / \text{ try to find } \boxed{P(z) = \sum_{i=0}^{\infty} z^i p_i}$$

$$\lambda \left(z^2 \sum_{k=0}^{\infty} z^k p_k + z\left(\sum_{k=0}^{\infty} z^k p_k - p_0\right)\right) = \mu \left(\sum_{k=0}^{\infty} z^k p_k - z p_1 - p_0 \right)$$

$$\lambda z \left(z P(z) + P(z) - p_0 \right) = \mu \left(P(z) - z p_1 - p_0 \right)$$

$$P(z)(\lambda z^2 + \lambda z - \mu) = \lambda z p_0 - \cancel{\mu z} p_1 - \mu p_0 \qquad / \frac{1}{\mu}$$

$$P(z) = \frac{p_0 + \cancel{z p_1} - \cancel{\rho z p_0}}{1 - \rho z - \rho z^2} \qquad p_0 = ? \quad p_1 = ?$$

use: $p_1 = \rho p_0 \Rightarrow$

$$P(z) = \frac{p_0}{1 - \rho z - \rho z^2}$$

use $\displaystyle \sum_{k=0}^{\infty} p_k \cdot z^k \Big|_{z=1} = \sum_{k=0}^{\infty} p_k = 1 \Rightarrow$

$$1 = \frac{p_0}{1 - \rho - \rho} \Rightarrow p_0 = 1 - 2\rho \Rightarrow$$

$$P(z) = \frac{1 - 2\rho}{1 - \rho z - \rho z^2}$$

$$\bar{N} = \frac{dP(z)}{dz}\Big|_{z=1} = \frac{(-1)(1-2\rho)(-\rho - 2\rho z)}{(1 - \rho z - \rho z^2)^2}\Big|_{z=1} = \frac{-1\cancel{(1-2\rho)}(-3\rho)}{(1-2\rho)^2} = \frac{3\rho}{1-2\rho} =$$

$$= \underline{\underline{\frac{3\lambda}{\mu - 2\lambda}}}$$

3.3

Holding time, T: $f(t) = \mu e^{-\mu t}$ $\quad E[T] = \int_0^\infty t\, f(t)\, dt = \int_0^\infty \mu t e^{-\mu t} dt = \frac{1}{\mu}$

Arrival Process $\quad P[N_c = k \mid T = t] = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad E[N_c \mid t] = \lambda t$

$$P(N_c = k] = \int_0^\infty P[N_c = k \mid T = t]\, f(t)\, dt$$

$$\underline{\underline{E[N_c]}} = \sum_{k=0}^\infty k \cdot P[N_c = k] =$$

$$\sum_{k=0}^\infty k \cdot \int_0^\infty P[N_c = k \mid T = t]\, f(t)\, dt =$$

(finite)

$$\int_0^\infty \underbrace{\sum_{k=0}^\infty k\, P[N_c = k \mid T = t]}_{E[N_c \mid t] = \lambda t}\, f(t)\, dt =$$

$$\lambda \underbrace{\int_0^\infty t\, f(t)\, dt}_{E[T] = \frac{1}{\mu}} = \underline{\underline{\frac{\lambda}{\mu}}}$$