Institutionen för matematik, KTH

## Homework assignments to Lecture 3.

1. Prove that the family of functions 1, $\sqrt{2}\cos(2\pi nx)$, $n \geq 1$ and $\sqrt{2}\sin(2\pi nx)$, $n \geq 1$ is a unit-perpendicular family in $L^2(S^1)$. Derive that a complex Fourier series can also be written i a real form

$$f = \hat{f}_{ev}(0) + \sum_{n=1}^{\infty} \big(\hat{f}_{ev}(n)\sqrt{2}\cos(2\pi nx) + \hat{f}_{odd}(n)\sqrt{2}\sin(2\pi nx)\big),$$

where

$$\hat{f}_{ev}(0) = \int_0^1 f(y)\,dy;$$

$$\hat{f}_{ev}(n) = \sqrt{2}\int_0^1 f(y)\cos(2\pi ny)\,dy,\ \ n \geq 1$$

and similarly for $\hat{f}_{odd}(n)$.

2. Assume that a sequence of numbers $(x_n)_{n\geq 1}$ has a limit $\lim_{n\to\infty} x_n = y$. Prove that a sequence of arithmetic means

$$\sigma_N = \frac{1}{N}(x_0 + x_1 + \cdots + x_{N-1})$$

also has the same limit $y$.

3. Prove that for any $f$ in $L^2(S^1)$ we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) = \hat{f}(0) = \int_0^1 f,$$

where the limit is in sense of convergence in $L^2$.

Hint: compute Fourier coefficients of the sum and then use the Plancherel (Parseval) identity.