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# SF2705 Fourier Analysis

## Homework assignement for the Lecture 2

1. (4.2.6) Prove Minkowski's inequality $\|f+g\| \le \|f\|+\|g\|$, where $\|f\|$ is the norm of $f$ in the space $L^2(0,1)$.

2. (4.3.2) Assume that $(e_n)_{n\ge 1}$ is a unit-perpendicular family in $L^2(0,1)$ and that the sequence $c=(c_n)_{n\ge 1}$ is square summable i.e. $\sum_{n\ge 1} |c_n|^2 < \infty$. Prove that the series $\sum_{n\ge 1} c_n e_n$ converges in $L^2(0,1)$.

3. (4.3.4) Assume that $(e_n)_{n\ge 1}$ is a unit-perpendicular family in $L^2(0,1)$ and that $f \in L^2(0,1)$. We define $\hat{f}(n) = (f, e_n)$.

(1) Prove that

$$\left\| f - \sum_{k=1}^{n} \hat{f}(k) e_k \right\|^2 = \|f\|^2 - \sum_{k=1}^{n} |\hat{f}(k)|^2.$$

(2) Deduce Bessel's inequality $\sum_{k=1}^{n} |\hat{f}(k)|^2 \le \|f\|^2$.

(3) Prove that $(e_n)_{n\ge 1}$ is a unit-perpendicular basis if and only if the Plancherel identity holds for any $f$

$$\sum_{k=1}^{\infty} |\hat{f}(k)|^2 = \|f\|^2.$$

4. (4.3.5) Assume that $(e_n)_{n\ge 1}$ is a unit-perpendicular family in $L^2(0,1)$. Prove that this family spans the whole $L^2(0,1)$ if and only if the only function $g$ which is orthogonal to all $e_n$ is $g=0$. Hint: analyse $g = f - \sum_n \hat{f}(n) e_n$.