

## Recitation 2

- z and Laplace transforms, definition, properties
  - related recitation problems
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### Discrete distributions, z transform

X - discrete r.v., non-negative discrete values  $X \in \{0, 1, 2, \dots\}$

PMF:  $p_i = P(X=i)$

Generating (or z-) transform: (unilateral z transform ~~is~~ also  $z^i$ , not  $z^{-i}$ )

Def:  $G_X(z) = \sum_{i=0}^{\infty} p_i z^i$  (not:  $= E[z^X]$ )

Why?

- compact way of representing a distribution
- some operations, like convolution ( $\& P(X+Y)$ ) are simpler
- simplifies the solution of recursive equations.

### Properties

a) Linear property  $\underbrace{a f_k + b g_k}_{\text{Linear combination}} \Leftrightarrow a F(z) + b G(z)$

b) Convolution property

$H = X + Y$ ,  $\begin{matrix} X, Y \text{ iid r.v.} \\ \uparrow \quad \uparrow \\ f_X \quad g_Y \end{matrix} \Leftrightarrow H(z) = F(z)G(z)$

$$H(z) = \sum_{k=0}^{\infty} h_k z^k = \sum_{k=0}^{\infty} \underbrace{\sum_{i=0}^k f_i g_{k-i}}_{\text{convolution}} \cdot z^k =$$

$$\sum_{k=0}^{\infty} \sum_{i=0}^{\infty} f_i g_{k-i} z^k = \underbrace{\sum_{i=0}^{\infty} f_i z^i}_{F(z)} \underbrace{\sum_{k=i}^{\infty} g_{k-i} z^{k-i}}_{G(z)} = F(z)G(z)$$

c) Expectations Moments

$$G(1) = \sum_{i=0}^{\infty} p_i z^1 = \sum_{i=0}^{\infty} p_i = 1$$

$$E[X] = \frac{d}{dz} G(z) \Big|_{z=1} \quad \text{Proof: } \frac{d}{dz} G(z) \Big|_1 = \sum_{i=0}^{\infty} i \cdot p_i z^{i-1} \Big|_{z=1} = \sum_{i=0}^{\infty} i p_i = E[X]$$

$$E[X^2] = \frac{d^2}{dz^2} G(z) \Big|_{z=1} \quad \text{Proof: } \frac{d^2}{dz^2} G(z) \Big|_1 =$$

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$$E[X^2] = \left. \frac{d^2}{dz^2} G(z) \right|_{z=1} + \left. \frac{d}{dz} G(z) \right|_{z=1}$$

Proof:

$$= \left. \sum_{i=0}^{\infty} i(i-1) p_i z^{i-2} \right|_{z=1} + \left. \sum_{i=0}^{\infty} i \cdot p_i z^{i-1} \right|_{z=1} = \sum i^2 p_i - \sum i p_i + \sum i p_i = E[X^2]$$

$$\text{Var}[X] = E[X^2] - E[X]^2$$

1.3. Poisson distribution:  $P_k = P(X=k) = \frac{a^k}{k!} e^{-a}$   $k=0, 1, 2, \dots$   $a > 0$

a) Prove that  $\sum_{k=0}^{\infty} P_k = 1$  (That is, probability dist)

$$\sum_{k=0}^{\infty} P_k = \sum_{k=0}^{\infty} \frac{a^k}{k!} e^{-a} = e^a \cdot e^{-a} = 1 \quad \checkmark$$

b)  $\underline{P(z)} = \sum_{k=0}^{\infty} P_k \cdot z^k = \sum_{k=0}^{\infty} \left( \frac{a^k}{k!} e^{-a} \cdot z^k \right) = e^{z \cdot a} \cdot e^{-a} = \underline{\underline{e^{-a(1-z)}}}$  ] group

c)  $\underline{E[X]} = \left. \frac{d}{dz} P(z) \right|_{z=1} = \left. \frac{d}{dz} e^{-a(1-z)} \right|_{z=1} = a e^{-a(1-z)} \Big|_{z=1} = \underline{\underline{a}}$

$$\underline{E[X^2]} = \dots a^2 e^{-a(1-z)} + a e^{-a(1-z)} = \underline{\underline{a^2 + a}}$$

$$\underline{\underline{\text{Var}[X] = E[X^2] - E[X]^2 = a}}$$

Translate it to Poisson proc:

$$P_k = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad E[X] = \lambda t, \quad E[X^2] = (\lambda t)^2 + \lambda t \quad \text{Var}[X] = \lambda t$$

1.4. Poisson distribution, sum of random variables

$$X = \sum_{i=1}^n X_i, \quad X_i \sim \text{Poisson}(a_i)$$

$$G_X(z) = \prod_{i=1}^n G_{X_i}(z) = \prod_{i=1}^n e^{-a_i(1-z)} = e^{-\sum_{i=1}^n a_i(1-z)} \Leftrightarrow \text{Poisson}(\sum_{i=1}^n a_i)$$

## Continuous distributions, Laplace transform

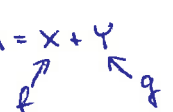
$X$ : non-negative cont. r.v. PDF  $f(x)$

Laplace transform:  $f^*(s)$ ,  $\mathcal{L}_x(s)$ ,  $\bar{F}^*(s)$

Def:  $f^*(s) = \int_0^{\infty} e^{-sx} f(x) dx$  ( $= E[e^{-sx}]$ )

### Properties

a) Linear property  $a f_x(x) + b g_x(x) \Leftrightarrow a f^*(s) + b g^*(s)$

b) Convolution:  $H = X + Y$   $h^*(s) = f^*(s) \cdot g^*(s)$  (independent!)  


c) Moments

$$E[X] = - \frac{d}{ds} f^*(s) \Big|_{s=0}$$

$$E[X^2] = \frac{d^2}{ds^2} f^*(s)$$

Proof:  $-\frac{d}{ds} \int_0^{\infty} e^{-sx} f(x) dx = - \int_0^{\infty} -x e^{-sx} f(x) dx$   
 $= E[-X e^{-sX}] \Big|_{s=0} = E[X]$

### 1.5 Exponential distribution

$$F(x) = P(X \leq x) = 1 - e^{-ax}, \quad x \geq 0$$

a)  $f(x) = \frac{dF(x)}{dx} = \frac{d}{dx} (1 - e^{-ax}) = a \cdot e^{-ax}$  (in class  $F(x) = 1 - e^{-\lambda x}$ ,  $f(x) = \lambda e^{-\lambda x}$ )

c)  $F^*(s) = \int_0^{\infty} e^{-sx} f(x) dx = \int_0^{\infty} a \cdot e^{-ax} e^{-sx} dx = a \int_0^{\infty} e^{-(s+a)x} dx = \frac{a}{s+a}$

d)  $E[X]$ ,  $E[X^2]$ ,  $\text{Var}[X]$  with Laplace transform

$$E[X] = - \frac{d}{ds} \frac{a}{s+a} \Big|_{s=0} = -a(-1) \frac{1}{(s+a)^2} \Big|_{s=0} = \frac{1}{a} \quad (\text{in class } E[X] = \frac{1}{\lambda})$$

$$E[X^2] = + \frac{d^2}{ds^2} \frac{a}{s+a} \Big|_{s=0} = \frac{a \cdot 2}{(s+a)^3} \Big|_{s=0} = \frac{2}{a^2} \quad (\text{in class } E[X^2] = \frac{2}{\lambda^2})$$

$$\text{Var}[X] = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

1.6 "Different events can happen, each within  $\text{Exp}(a)$  time..."

Is: course assignment

a)  $X = \min(x_1, \dots, x_n)$   $X_i = \text{Exp}(a)$  indep!

$$\bar{F}(x) = P(X > x) = P(\text{all } X_i > x) = \prod_{i=1}^n P(X_i > x) = e^{-\sum_{i=1}^n a_i x}$$

↑  
indep

Group work

$$F(x) = P(X \leq x) = 1 - e^{-\sum a_i x} \Rightarrow \text{Exp}(\sum a_i x)$$

1.7.  $X_i = \text{Exp}(a)$ , i.i.d.

a)  $X = \sum_{i=1}^r X_i \Rightarrow$  We work in transform domain

$$X_i \sim \text{Exp}(a) \Leftrightarrow f_{X_i}^*(s) = \frac{a}{s+a}$$

$$f_X^*(s) = \prod_{i=1}^r f_{X_i}^*(s) = \left(\frac{a}{s+a}\right)^r \cdot \text{Check Laplace transform table...}$$

$$\Rightarrow X \sim \text{Erlang-}r, f_X(x) = a \frac{(ax)^{r-1}}{(r-1)!} e^{-ax}$$

$$c) E[X] = \sum_{i=1}^r E[X_i] = \frac{r}{a}$$

$$\text{Var}[X] = \sum_{i=1}^r \text{Var}[X_i] = \frac{r}{a^2}$$

$$E[X^2] = \text{Var}[X] + E[X]^2 = \frac{r}{a^2} + \frac{r^2}{a^2} = \frac{r(r+1)}{a^2}$$

Home:  
Same work from  
 $f_X^*(s)$