EP2200 Home assignment I – Probability theory

1. Let A and B be two events such that:

$$P(A) = 0.4, P(B) = 0.7, P(A \cup B) = 0.9$$

(a) Find $P(A \cap B)$.
(b) Find $P(A^c \cap B)$.
(c) Find $P(A - B)$.
(d) Find $P(A^c - B)$.
(e) Find $P(A^c \cup B)$.
(f) Find $P(A \cap (B \cup A^c))$.

2. Suppose that of all the customers at a coffee shop:

-70% purchase a cup of coffee.
-40% purchase a piece of cake.
-20% purchase both a cup of coffee and a piece of cake.
Given that a randomly chosen customer has purchased a piece of cake, what is the probability that he/she has also purchased a cup of coffee?

3. One way to design a spam filter is to look at the words in an email. In particular, some words are more frequent in spam emails. Suppose that we have the following information:

   (a) 50% of emails are spam.

   (b) 1% of spam emails contain the word "refinance".

   (c) 0.001% of non-spam emails contain the word "refinance".

   Suppose that an email is checked and found out to contain the word refinance. What is the probability that the email is a spam?

4. Let X and Y be two independent discrete random variables with the following PMFs:

$$P_X(k) = \begin{cases} \frac{1}{4} & \text{for } k = 1 \\ \frac{1}{8} & \text{for } k = 2 \\ \frac{1}{8} & \text{for } k = 3 \\ \frac{1}{2} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$P_Y(k) = \begin{cases} \frac{1}{6} & \text{for } k = 1 \\ \frac{1}{6} & \text{for } k = 2 \\ \frac{1}{3} & \text{for } k = 3 \\ \frac{1}{3} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $P(X \leq 2 \text{ and } Y \leq 2)$.
(b) Find $P(X > 2 \text{ or } Y > 2)$.
(c) Find $P(X > 2 | Y > 2)$.
(d) Find $P(X < Y)$.

5. Suppose that $Y = -2X + 3$. If we know $E[Y] = 1$ and $E[Y^2] = 9$, find $E[X]$ and $\text{Var}(X)$.

6. Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} ce^{-4x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is a positive constant.

(a) Find c.
(b) Find the CDF of X, $F_X(x)$.
(c) Find $P(2 < X < 5)$.
(d) Find $E[X]$.

7. Prove the following useful properties of random variables X and Y:

$$E[cX] = cE[X]$$

$$E[X + Y] = E[X] + E[Y]$$

$$\text{Var}[cX] = c^2 \text{Var}[X]$$

If X and Y are independent, then

$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$$

Consider the mixture distribution given by pdf $p(x) = a_1 p_1(x) + a_2 p_2(x)$, $a_1 + a_2 = 1$. Show that

$$E[X] = a_1 E_{p_1}[X] + a_2 E_{p_2}[X]$$

$$E[X^2] = a_1 E_{p_1}[X^2] + a_2 E_{p_2}[X^2]$$