Lösningar Reglerteknik AK Tentamen 2015–10–30

Uppgift 1a

Systemet är stabilt (pol i −2), så vi kan använda slutvärdesteoremet för att bestämma

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s)\frac{10}{s} = G(0)10 = 5l_0 = r = 10$$

Svar: $l_0 = 2$

Uppgift 1b

Svar: Insignal: flöde av distillerat vatten som styrs via ventilen.
Utsignal: saltkoncentrationen i utflödet.
Störsignal: Variationer i saltkoncentration i koksaltlösningen.

Uppgift 1c

Känslighetsfunktionen är stabil så vi kan använda frekvensanalys för att räkna ut störningsundertryckningen

$$|S(i0.2)| = \left|\frac{0.2i}{0.2i+10}\right| = \frac{0.2}{\sqrt{100+0.2^2}} < \frac{1}{50}$$

Svar: En sinus-störning med frekvens 0.2 radianer per sekund undertrycks mer än med en faktor femtio, dvs specifikationen är uppfylld.

Uppgift 1d

Det finns flera möjliga lösningar. Vi använder här diagonalform via partialbråksuppdelning

$$G(s) = \frac{10}{(s+2)(s-1)(s+4)} = \frac{a}{(s+2)} + \frac{b}{(s-1)} + \frac{c}{(s+4)}$$

där $a = -5/3,\ b = 2/3\ c = 1$. Detta innebär att

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -5/3 & 2/3 & 1 \end{bmatrix}$$

Svar: Egenvärdena till A är lika med polerna till $G(s)$, d.v.s. $-2, 1, -4$. Detta kan direkt ses via diagonala valet av A

Uppgift 2a

Från figuren ses att polerna ligger i $\{-1, -\varepsilon + i, -\varepsilon - i\}$. Ansätt därför att

$$\begin{aligned} H(s) &= \frac{1}{s-(-1)} \times \frac{1}{s-(-\varepsilon+i)} \times \frac{1}{s-(-\varepsilon-i)} \\ &= \frac{1}{s+1} \times \frac{1}{(s+\varepsilon-i)(s+\varepsilon+i)} \\ &= \frac{1}{s+1} \times \frac{1}{(s+\varepsilon)^2 + (s+\varepsilon)i - i(s+\varepsilon) - i^2} \\ &= \frac{1}{s+1} \times \frac{1}{(s+\varepsilon)^2+1}. \end{aligned}$$

Denna överföringsfunktion har dock statisk förstärkning

$$H(0) = \frac{1}{0+1} \times \frac{1}{(0+\varepsilon)^2+1} = \frac{1}{\varepsilon^2+1}.$$

Den sökta överföringsfunktionen $G(s)$ ska ha statisk förstärkning ett, så vi normaliserar $H(s)$ enligt

$$G(s) = \frac{H(s)}{H(0)} = \frac{1}{s+1} \times \frac{\varepsilon^2+1}{(s+\varepsilon)^2+1},$$

så att $G(0) = 1$.

Svar: $G(s) = \frac{1}{s+1}\frac{\varepsilon^2+1}{(s+\varepsilon)^2+1}$.

Uppgift 2b

För stora ε är den reella polen klart dominant. Så stegsvar C hör till $\varepsilon = \frac{3}{2}$. Dämpningen (cosinus av vinkeln mot negativa reella axeln) minskar allteftersom de komplexa polerna rör sig mot den imaginära axeln, dvs. när ε minskar. Detta ger att D hör till $\varepsilon = \frac{1}{3}$, B till $\varepsilon = \frac{1}{2}$ och A till $\varepsilon = \frac{2}{3}$.

Svar: $\varepsilon_A = \frac{2}{3}, \varepsilon_B = \frac{1}{2}, \varepsilon_C = \frac{3}{2}$ och $\varepsilon_D = \frac{1}{3}$.

Uppgift 2c

Eftersom $G_1(s) = sG(s)$ får vi

$$|G_1(i\omega)| = \omega|G(i\omega)| \quad \Rightarrow \quad \begin{aligned} \log|G_1(i\omega)| &= \log\omega + \log|G(i\omega)| \\ \arg G_1(i\omega) &= \frac{\pi}{2} + \arg G(i\omega) \end{aligned}$$

Detta innebär att förstärkningen (amplitudkurvan) skalas med ω vilket i logaritmisk skala motsvaras av addition med $\log\omega$ (extra lutning $+1$ för alla frekvenser)
Faskurvan ökas med $\pi/2 = 90^o$ för all frekvenser.

Uppgift 3a

All the values can be found in the Nyquist plot:

- $\omega_c = 5.28$ rad/s;
- $\omega_p = 12.6$ rad/s;
- $\phi_m = \arctan(\frac{0.74}{0.85}) = 41.04^o$;
- $A_m = \frac{1}{0.21} = 4.76$.

Uppgift 3b

We design a lead-lag controller

$$F(s) = K\frac{\tau_D s + 1}{\beta\tau_D s + 1}\frac{\tau_I s + 1}{\tau_I s + \gamma}$$

Thus, we need to determine all the parameters of the controller such that the closed loop system fulfills the problem requirements.
The desired phase margin of at least $\phi_m = 60^o$ means that we need to increase the systems phase by $\phi_{max} = 60^o - 41.04^o + 5.7^o = 24.66^o$, where 5.7^o is the phase decrease amount due to the lag part. The second requirement means that the crossover frequency must be $\omega_{c,d} \approx \omega_B \approx \omega_c = 5.28$ rad/s. Finally, the last requirement yields $\gamma = 0$, since (for stable system)

$$Y(s) = \frac{G(s)}{1 + G(s)F(s)}L(s) \;\Rightarrow\; \lim_{t\to\infty} y(t) = \frac{G(0)}{1 + G(0)F(0)} = 0$$

if and only if $F(s)$ contains an integrator.
Therefore,

- $\phi_{max} \Rightarrow \beta = 0.4$ (Fig. 5.13, p. 106, Glad and Ljung book);
- $\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 0.30$;
- $K = \frac{\sqrt{\beta}}{|G(j\omega_{c,d})|} = 0.63$;
- $\tau_I = \frac{10}{\omega_{c,d}} = 1.90$.

Answer:

$$F(s) = K\frac{\tau_D s + 1}{\beta\tau_D s + 1}\frac{\tau_I s + 1}{\tau_I s + \gamma}, \quad \beta = 0.4, \tau_D = 0.30, K = 0.63, \tau_I = 1.90$$

Uppgift 3c

The time delay only affects the phase of the open loop system. Thus, $\arg(F(j\omega)G(j\omega)e^{-j\omega T}) = arg(F(j\omega)G(j\omega)) - \omega T$. This means that the phase will shift ωT rad. With an allowed phase shift of 15^o , we have $\omega_c T \leq \frac{15\pi}{180}$ with $\omega_c = 5.28$ rad/s **Answer:** $T \leq 0.05$ s

Uppgift 4

(a) The closed loop transfer function is given by

$$G_c(s) = \frac{G(s)F(s)}{1+G(s)F(s)} = \frac{2K_1(s+2)}{s(s+3)+2K_1(s+2)}.$$

From its denominator, we identify:

$$\begin{aligned} P(s) &= s(s+3), \\ Q(s) &= 2(s+2). \end{aligned}$$

so the root locus has $n = 2$ start points, ($p_1 = 0$ and $p_2 = -3$), $m = 1$ end point ($q_1 = -2$), and $n-m = 1$ asymptote. We calculate the intersection with the imaginary axis:

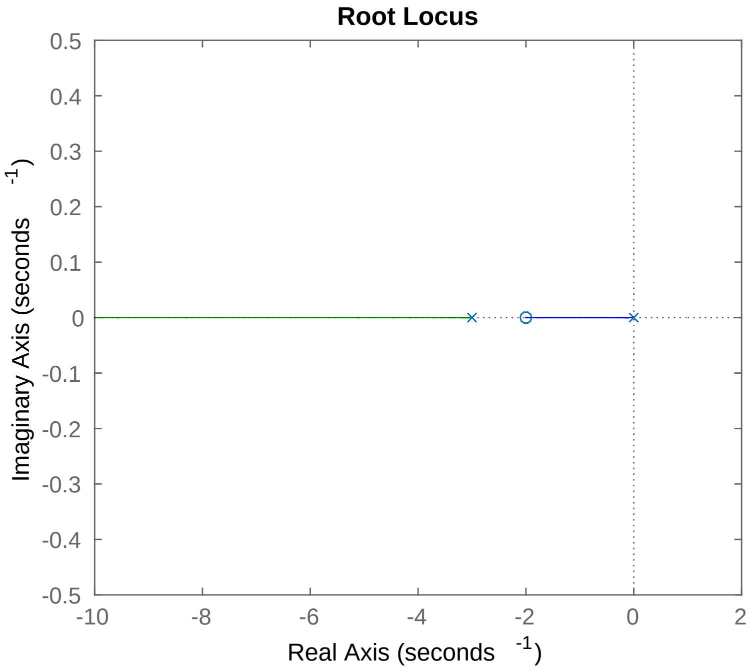
$$\begin{gathered} P(j\omega^*) + K_1 Q(j\omega^*) = 0 \\ j\omega^*(j\omega^* + 3) + 2K_1(j\omega^* + 2) = 0 \\ -\omega^{*2} + 3j\omega^* + 2jK_1\omega^* + 4K_1 = 0 \end{gathered}$$

which gives the systme of equation*s

$$\begin{cases} -\omega^{*2} + 4K_i = 0 \\ 3\omega^* + 2K_i\omega^* = 0 \end{cases}$$

that has the solution $\omega^* = 0$ and $K_1 = 0$. This indicates that there are no crossings except the stating point.

Below is a plot of the resulting root locus



As the root locus is contained in the left half plane, the system is stable for all $K_1 > 0$. QED

(b) When $K_1 = 1$ the closed loop transfer function becomes

$$G_c(s) = \frac{2s+4}{s^2+5s+4}.$$

The reference signal is an unit step, so

$$R(s) = \frac{1}{s}$$

The system output, in the Laplace domain, is given by

$$Y(s) = W(s)R(s) = \frac{2s+4}{s^2+5s+4}\frac{1}{s} = \frac{2s+4}{s^3+5s^2+4s} = \frac{2s+4}{s(s+1)(s+4)}$$

Using the partial fraction expansion, we have that

$$Y(s) = \frac{1}{s} - \frac{\frac{2}{3}}{s+1} - \frac{\frac{1}{3}}{s+4}.$$

Applying the inverse Laplace transform, we have in the time domain:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{Y(s)\right\}[t] = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}[t] - \mathcal{L}^{-1}\left\{\frac{\frac{2}{3}}{s+1}\right\}[t] - \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s+4}\right\}[t], \\ &= h(t) - \frac{2}{3}e^{-t}h(t) - \frac{1}{3}e^{-4t}h(t). \end{aligned}$$

where $h(t)$ is the Heavyside step function.

Since $r(t) = h(t)$, we can easily calculate

$$\lim_{t\to\infty}[r(t) - y(t)] = \lim_{t\to\infty} \frac{2}{3}e^{-t}h(t) + \frac{1}{3}e^{-4t}h(t).$$

(c) The closed loop transfer function is given by

$$G_c(s) = \frac{G(s)F(s)}{1 + G(s)F(s)} = \frac{K_2\omega_0(s+2)}{(s^2+\omega_0^2)(s+3) + (s+2)K_2\omega_0}.$$

From its denominator, we identify:

$$\begin{aligned} P(s) &= (s^2+\omega_0^2)(s+3), \\ Q(s) &= (s+2)\omega_0. \end{aligned}$$

so the root locus has $n = 3$ start points ($p_1 = -3$ and $p_{2,3} = \pm\omega_0$), $m = 1$ end point ($q_1 = -2$), and $n - m = 2$ asymptotes, with intersect

$$z = \frac{\sum_i p_i - \sum_i q_i}{n - m} = \frac{\omega_0 + (-\omega_0) + (-3) - (-2)}{2} = -\frac{1}{2}.$$

We calculate the intersection with the imaginary axis:

$$\begin{gathered} P(j\omega^*) + K^*Q(j\omega^*) = 0 \\ \big((j\omega^*)^2 + \omega_0^2\big)(j\omega^* + 3) + K^*\omega_0(j\omega^* + 2) = 0 \\ \big(-\omega^{*2} + \omega_0^2\big)(j\omega^* + 3) + K^*\omega_0(j\omega^* + 2) = 0 \\ -j\omega^{*3} - 3\omega^{*2} + j\omega_0^2\omega^* + 3\omega_0^2 + jK^*\omega_0\omega^* + 2K^*\omega_0 = 0 \end{gathered}$$

which gives the system of equation*s

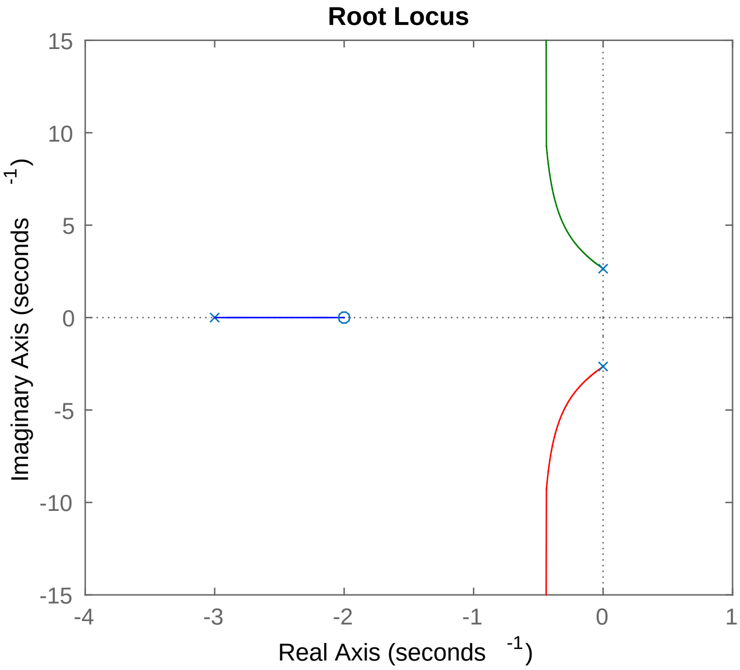
$$\begin{cases} -\omega^{*3} + \omega_0^2\omega^* + K^*\omega_0\omega^* = 0 \\ -3\omega^{*2} + 3\omega_0^2 + 2K^*\omega_0 = 0 \end{cases}$$

that has the solutions:

$$\begin{gathered} \omega_1^* = 0, \quad K_1^* = \frac{-3}{2} \\ \omega_{2,3}^* = \pm\omega_0, \quad K_{2,3}^* = 0 \end{gathered}$$

The first solution is discarded as it corresponds to a negative gain, the other two indicate starting points, so there are no crossings except for the starting points.

Below is a plot of the resulting root locus



As the root locus is contained in the left half plane, the system is stable for all $K_2 > 0$. QED

(e) If $r(t) = \sin(\omega_0 t)$, frequency analysis tells us that the response will be, at steady state

$$y_{ss}(t) = |W(j\omega_0)| \sin(\omega_0 t + \angle W(j\omega_0)).$$

Since

$$G_c(j\omega_0) = \frac{(j\omega_0 + 2)K\omega_0}{\big((j\omega_0)^2 + w_0^2\big)(s+3) + (j\omega_0 + 2)K\omega_0} = 1,$$

we have

$$|G_c(j\omega_0)| = 1, \qquad \angle W(j\omega_0) = 0,$$

and

$$y_{ss}(t) = \sin(\omega_0 t).$$

At steady state the error is thus

$$r(t) - y(t) = 0,$$

and the proposed controller does solve the problem of reference tracking with zero error.

Exercise 5

(a) After closing the loop, the equation*s in state space form become

$$\begin{aligned} \dot{x}(t) &= (a-l)x(t), \qquad x(0) = x_0, &(1)\\ y(t) &= x(t). &(2) \end{aligned}$$

Substituting $y(t) = x(t)$, we have that $y(t)$ is the solution to the first order ODE

$$\dot{y}(t) = (a-l)y(t),$$

which means

$$y(t) = \kappa e^{(a-l)t},$$

where κ is a constant that depends on the initial conditions, in this case

$$y(0) = \kappa = x(0) = x_0,$$

and the solution is thus

$$y(t) = x_0 e^{(a-l)t}$$

The corresponding input signal is given by

$$u(t) = -lx(t) = -ly(t) = -lx_0 e^{(a-l)}t.$$

The closed loop system has dynamic matrix $A = a - l$, that has one eigenvalue in $a - l$, so the closed loop system is stable as long as

$$\mathbf{Re}\{a-l\} = a - l < 0$$

or

$$l > a.$$

(b) We calculate the quadratic criterion

$$J(l) = \int_0^\infty [y(\tau)^2 + qu(\tau)^2] \mathrm{d}\tau.$$

Plugging in the solutions from point (a), we have that

$$\begin{aligned} J(l) &= \int_0^\infty \left[\left(x_0 e^{(a-l)\tau}\right)^2 + q\left(-lx_0 e^{(a-l)\tau}\right)^2 \right] \mathrm{d}\tau \\ &= \int_0^\infty \left[x_0^2 e^{2(a-l)\tau} + ql^2 x_0^2 e^{2(a-l)\tau}\right] \mathrm{d}\tau \\ &= \int_0^\infty x_0^2(1+ql^2) e^{2(a-l)\tau} \mathrm{d}\tau \\ &= \left| x_0^2(1+ql^2)\frac{e^{2(a-l)\tau}}{2(a-l)} \right|_0^\infty \\ &= -\frac{x_0^2(1+ql^2)}{2(a-l)} \end{aligned}$$

To find the optimal value for l, we find the l such that

$$\frac{\mathrm{d}}{\mathrm{d}l}J(l) = 0,$$

which means

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}l}J(l) &= 0, &(3)\\
-\frac{\mathrm{d}}{\mathrm{d}l}\frac{x_0^2(1+ql^2)}{2(a-l)} &= 0, &(4)\\
-\frac{x_0^2}{2}\frac{\mathrm{d}}{\mathrm{d}l}\frac{(1+ql^2)}{a-l} &= 0, &(5)\\
-\frac{2ql(a-l)+(1+ql^2)}{(a-l)^2} &= 0, &(6)\\
2ql(a-l)+1+ql^2 &= 0, &(7)\\
l^2 - 2al - \frac{1}{q} &= 0. &(8)\\
& &(9)
\end{aligned}$$

This second order equation* has the solutions

$$l_{1,2} = a \pm \sqrt{a^2 + \frac{1}{q}}$$

of which

$$l_1 = a + \sqrt{a^2 + \frac{1}{q}}$$

is the stabilizing solution, since $l_1 > a$ according to point (a).

(c) Using the optimal LQ regulator $u(t) = -l_1 x(t)$, we have that the system pole is given (in general as a function of q) by

$$p(q) = a - l_1(q) = -\sqrt{a^2 + \frac{1}{q}}.$$

When $q \to \infty$, we require small input signals, irrespective of the output convergence speed. In this case, the feedback controller gain is

$$l_1(0) = a + |a|.$$

If the system is stable, $a < 0$ and $l_1(0) = 0$, effectively running the system in open loop with zero input, and the closed loop pole becomes

$$p(0) = a - l_1(0) = a.$$

If the system is unstable, $a > 0$ and the controller mirrors the pole in the negative half plane:

$$p(0) = a - l_1(0) = -a.$$