# Solutions, Tutorial 4

**1.**

a)

The maximum height of a Chapman layer is given by (Equation 3.5.8 in Fälthammar)

$$z_{\max} = H \ln\left(a_a n_0 H\right)$$

where the scale height $H$ is given by

$$H = \frac{k_B T}{gm}$$

The pressure of the neutral atmosphere at the Jupiter surface is given by

$$p = n_0 k_B T \quad \Rightarrow$$

$$n_0 = \frac{p}{k_B T} = \frac{100 \cdot 10^3}{1.38 \cdot 10^{-23} \cdot 300} = 2.4 \cdot 10^{25}\,\mathrm{m}^{-3}$$

<u>For atomic hydrogen:</u>

Here we use the mass of the atomic hydrogen in the expression for the scale height and the appropriate value for the absorption coefficient, $\bar{\sigma}\left(H\right)$.

$$H = \frac{k_B T}{gm} = \frac{1.38 \cdot 10^{-23} \cdot 300}{1.67 \cdot 10^{-27} \cdot 23.1} = 107 \text{ km}$$

$$z_{\max} = H \ln\left(a_a n_0 H\right) = 107 \cdot \ln\left(3 \cdot 10^{-18} \cdot 10^{-4} \cdot 2.4 \cdot 10^{-25} \cdot 107 \cdot 10^3\right) = 1700 \text{ km}$$

<u>For molecular hydrogen:</u>

Here we use the mass of the molecular hydrogen in the expression for the scale height and the appropriate value for the absorption coefficient $\bar{\sigma}\left(H_2\right)$.

$$H = \frac{k_B T}{gm} = \frac{1.38 \cdot 10^{-23} \cdot 300}{2 \cdot 1.67 \cdot 10^{-27} \cdot 23.1} = 54 \text{ km}$$

$$z_{\max} = H \ln\left(a_a n_0 H\right) = 54 \cdot \ln\left(6.1 \cdot 10^{-18} \cdot 10^{-4} \cdot 2.4 \cdot 10^{25} \cdot 54 \cdot 10^{3}\right) = 819 \text{ km}$$

Atomic hydrogen gives the more realistic altitude of the electron density peak.

b)

$$\frac{dn_e}{dt} = q - \alpha n_e^2$$

$$q = 0 \quad \Rightarrow$$

$$\frac{dn_e}{dt} = -\alpha n_e^2 \qquad \Rightarrow$$

$$\int_{n_{e0}}^{n_e} \frac{dn_e}{n_e^2} = -\alpha \int_{t_0}^{t} dt \qquad \Rightarrow$$

$$\left[-\frac{1}{n_e}\right]_{n_{e0}}^{n_e} = \alpha\left(t - t_0\right) \quad \Rightarrow$$

$$\frac{1}{n_e} = \frac{1}{n_{e0}} + \alpha\left(t_0 - t\right) \quad \Rightarrow$$

$$n_e = \frac{1}{\dfrac{1}{n_{e0}} + \alpha\left(t_0 - t\right)} = \frac{n_{e0}}{1 + \alpha n_{e0}\left(t_0 - t\right)}$$

Jupiter’s rotation period is 0.41·24 h = 35424 s.

<u>For atomic hydrogen:</u>

$n_{e0} = 2.5 \cdot 10^5$ cm$^{-3}$

$\alpha = 10^{-12}$ cm$^3$s$^{-1}$

Thus

$$n_e = \frac{2.5 \cdot 10^5}{1 + 10^{-12} \cdot 2.5 \cdot 10^5 \cdot \dfrac{35424}{2}} = 2.49 \cdot 10^5 \ \text{cm}^{-3}$$

Virtually unchanged!

<u>For molecular hydrogen:</u>

$n_{e0} = 2.5\cdot10^5$ cm$^{-3}$

$\alpha$ = $10^{-8}$ cm$^3$s$^{-1}$

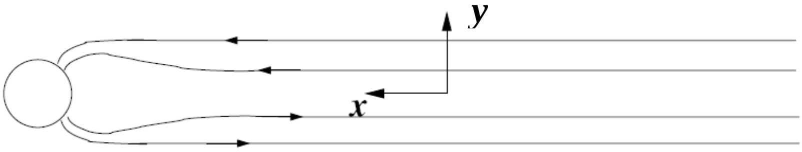
Thus

$$n_e = \frac{2.5\cdot10^5}{1+10^{-8}\cdot2.5\cdot10^5\cdot\frac{35424}{2}} = 5.5\cdot10^3\,\mathrm{cm}^{-3}$$

Changes by a factor of 45.

**2.**

$$\mathbf{B} = \begin{cases} -B_0\hat{\mathbf{x}} & , \quad y < -a \\ B_0\hat{\mathbf{x}}\dfrac{3a^2y - y^3}{2a^3} & , \quad -a \le y \le a \\ B_0\hat{\mathbf{x}} & , \quad y > a \end{cases}$$




**a)**

$$\mathbf{j} = -\frac{1}{\mu_0}\frac{\partial B_x}{\partial y}\hat{\mathbf{z}} = \hat{\mathbf{z}}\begin{cases} 0 & , \quad y < -a \\ -\dfrac{1}{\mu_0}B_0\dfrac{3a^2 - 3y^2}{2a^3} & , \quad -a \le y \le a \\ 0 & , \quad y > a \end{cases}$$

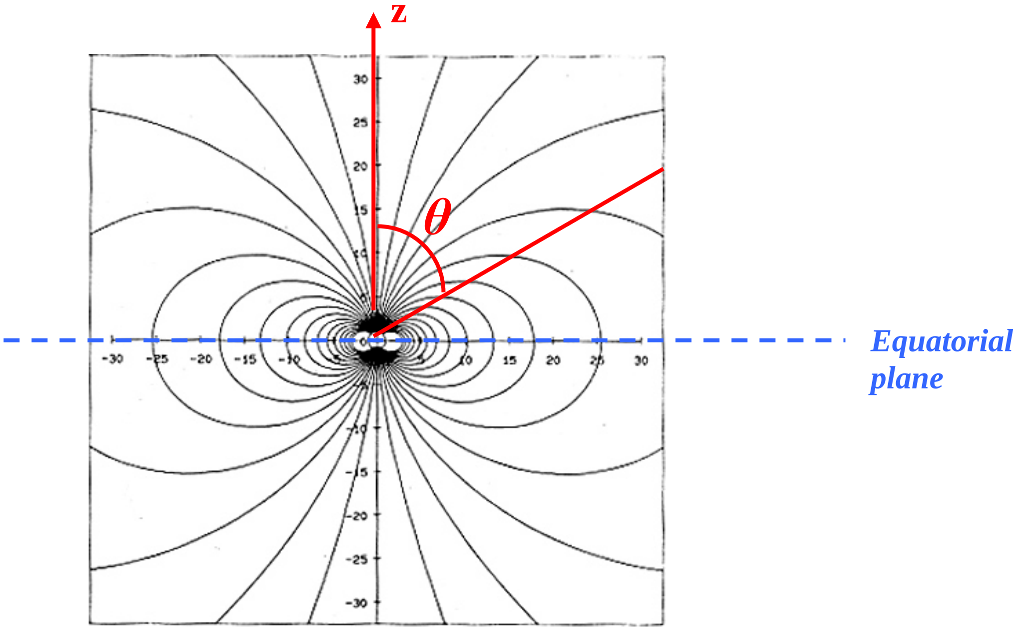
For $y$ = 0:

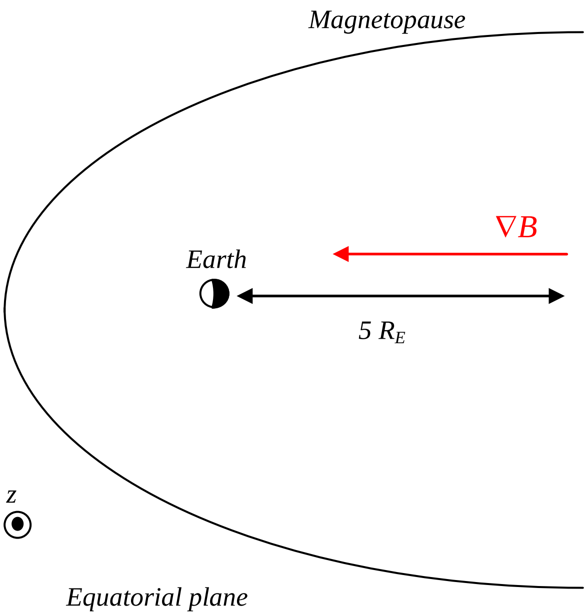
$$j_z(x,0) = -\frac{1}{\mu_0}B_0\frac{3a^2}{2a^3} = -\frac{3}{2}\frac{B_0}{\mu_0 a} = -\frac{3}{2}\frac{10^{-8}}{4\pi\cdot10^{-7}\cdot2\cdot10^6}\,\mathrm{Am}^{-2} = 6.0\cdot10^{-9}\,\mathrm{Am}^{-2}$$

**b)**

$$I = l\int_{-a}^{a}|\mathbf{j}|dy = \frac{l}{\mu_0}\int_{-a}^{a}\left|\frac{\partial B_x}{\partial y}\right|dy = \frac{l}{\mu_0}|B_x(a) - B_x(-a)| = \frac{80\cdot 6378\cdot 10^3}{\mu_0}\left(20\cdot 10^{-9}\right) = 8.1\,\mathrm{MA}$$

**3.**




$$B^2 = B_P{}^2 R_E{}^6 r^{-6}\cos^2\theta + \frac{B_P{}^2}{4} R_E{}^6 r^{-6}\sin^2\theta$$

What is $\nabla B$? It is actually easier to calculate $\nabla B^2 = 2B\nabla B$. In polar coordinates

$$\left(\nabla B^2\right)_r = \frac{\partial B^2}{\partial r} = B_P{}^2 R_E^6(-6r^{-7}\cos^2\theta - \frac{6}{4}r^{-7}\sin^2\theta)$$

$$\theta = 90^\circ \quad \Rightarrow$$

$$\left(\nabla B^2\right)_r = -\frac{3}{2}B_P{}^2 R_E^3 r^{-7}$$

The $\theta$ component is zero:

$$\left(\nabla B^2\right)_\theta = \frac{1}{r}\frac{\partial B^2}{\partial\theta} = \frac{1}{r}B_P{}^2 R_E^6 r^{-6}(-2\cos\theta\sin\theta + \frac{1}{4}2\cos\theta\sin\theta)$$

$$\theta = 90^\circ \quad \Rightarrow$$

$$\left(\nabla B^2\right)_\theta = 0$$

Also

$$\theta = 90^\circ \quad \Rightarrow$$

$$B = \frac{1}{2}B_P R_E^3 r^{-3}$$

Then

$$\nabla B = \hat{\mathbf{r}}\frac{1}{2B}\left(\nabla B^2\right)_r = -\frac{3}{2}\frac{B_P{}^2 R_E^6 r^{-7} 2}{2B_P R_E^3 r^{-3}}\hat{\mathbf{r}} = -\frac{3}{2}B_P R_E^3 r^{-4}\hat{\mathbf{r}}$$

i.e. pointing towards Earth..

The drift velocity is

$$\mathbf{v}_{drift} = \frac{\mathbf{F}\times\mathbf{B}}{qB^2} = -\frac{\mu\left(\nabla B\right)\times\mathbf{B}}{qB^2}$$

Since all angles are 90 degrees

$$v_{drift} = \left|\mathbf{v}_{drift}\right| = \frac{\mu\left|\nabla B\right|}{qB}$$

Further

$$\mu = \frac{mv_\perp^2}{2B} = \frac{mv^2}{2B} = \frac{W}{B} \quad \text{since}$$

$$\alpha = 90° \quad \Rightarrow \quad v_\perp = v$$

Thus

$$v_{drift} = \frac{W}{B}\frac{|\nabla B|}{qB} = \frac{W|\nabla B|}{qB^2} = \frac{W \cdot 3B_P R_E^3 \cdot 2^2 r^6}{q \cdot 2r^4 B_P^2 R_E^6} = \{r = 5R_E\} = \frac{6 \cdot 25 \cdot W}{qB_P R_E} = 0.379\frac{W}{q}$$

For electrons with W = $10^4$ eV = $10^4$q J we get

$$v_{drift} = \frac{W}{B}\frac{|\nabla B|}{qB} = 0.379\frac{10^4 q}{q} = 3790\,\text{m/s}$$

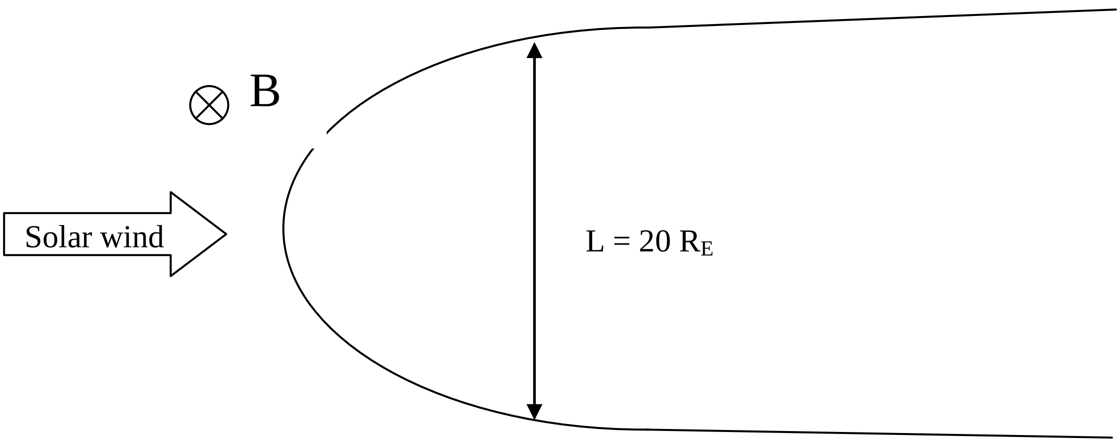
The revolution time $T$ is

$$T = \frac{O}{v} = \frac{2\pi \cdot 5R_E}{v} = 14.7\,\text{h}$$

For the ions T will be 16.8 years.

**4.**

Seeing the magnetosphere from ”above”:




The induced electric field from the solar wind is

$$\mathbf{E} = -\mathbf{v}_{SW} \times \mathbf{B}_{SW}$$

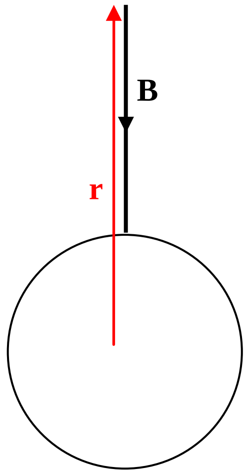
If we just care about the magnitude

$$E = vB$$

This gives a potential drop in the east-west direction over the magnetosphere of

$$EL = vBL = 350 \cdot 10^3 \cdot 7 \cdot 10^{-9} \cdot 20 \cdot 6378 \cdot 10^3 \approx 310 \text{ kV}$$

**5.**




The distance $r$ from Earth's centre is 10 000 km + 1 $R_E$ = 16378 km. With $\theta = 0$ we get

$$B(r) = \frac{\mu_0 a}{2\pi} \frac{1}{r^3}$$

The electron is mirrored when the magnetic field is

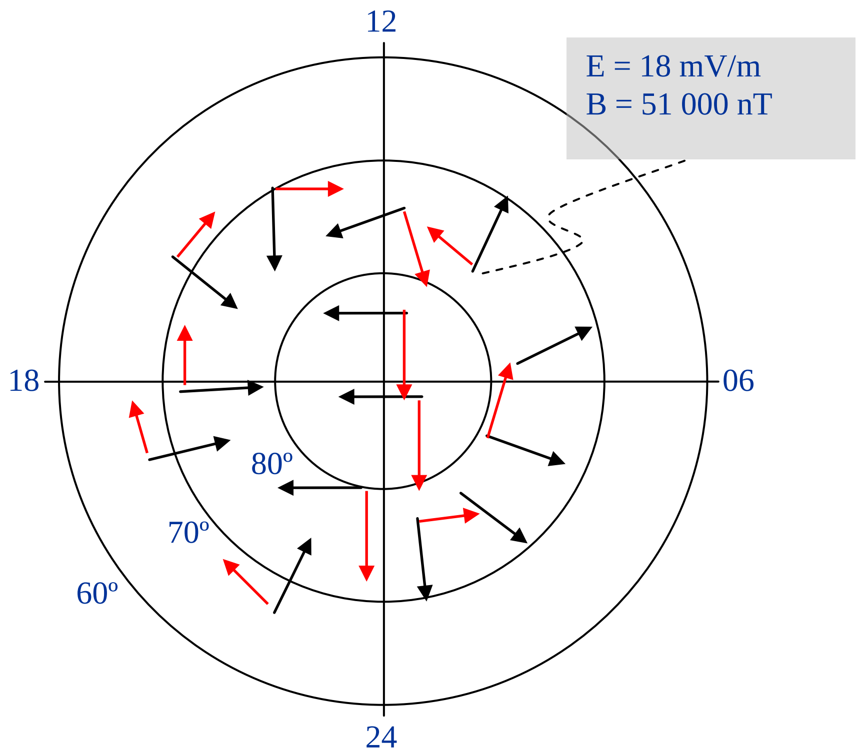
$$B_{turn} = \frac{B}{\sin^2 \alpha} \quad \Rightarrow$$

$$\frac{\mu_0 a}{2\pi} \frac{1}{r_{turn}^3} = \frac{\mu_0 a}{2\pi} \frac{1}{r^3 \sin^2 \alpha} \quad \Rightarrow$$

$$r_{turn} = r(\sin \alpha)^{2/3} = 16378 \cdot (\sin 15°)^{2/3} = 6655\,\text{km}$$

The altitude h will then be h = 6655 – 6378 = 273 km. This means that this electron will have a reasonable chance to collide with a neutral atom or molecule to produce aurora.

**6.**




$$v_d = \frac{\mathbf{E}\times\mathbf{B}}{B^2} = \frac{E}{B} = \frac{18\cdot 10^{-3}}{51000\cdot 10^{-9}} = 353\ \mathrm{ms}^{-1}$$