Homework 5
FD-TD method for the Maxwell equations
Part 1-2 (Mandatory) max 4.0 p
Part 3 (Optional) max 4.0 p to be counted on the exam.
Deadline Mon May 25, 2015

The aim of this exercise is to experiment with the FD-TD method for the Maxwell equations, see e.g. the CEMbook, and the classic:

TafloveFDTD: Computational Electrodynamics: The Finite-Difference Time-Domain Method, second edition, A. Taflove and S. Hagness, Artech House 2000.
TafloveFDTD has the 2D FDTD MATLAB code used for the 2D modeling experiments. The m-file can be downloaded from the course homepage.

1 The Maxwell equations in 1D

The Maxwell equations in 3D for a plane wave propagating in the x-direction with a polarization such that $E_y = 0$, $E_z \neq 0$ are

$$\begin{cases} Hy_t = \dfrac{1}{\mu} Ez_x - \dfrac{\rho}{\mu} Hy \\ Ez_t = \dfrac{1}{\varepsilon} Hy_x - \dfrac{\sigma}{\varepsilon} Ez \end{cases} \tag{1}$$

where the three components of the electric field **E** are (Ex, Ey, Ez) and the three components of the magnetic field **H** are (Hx, Hy, Hz).

2 General implementation in 1D

Below we give a step by step description for adding functionality to your 1D code. It is highly recommended that you implement all the steps in one Matlab code by using sub-functions or sub-programs. It shall be possible to control the code behavior by control variables or input parameters. Implement the Yee scheme in vacuum for (1) on this grid:
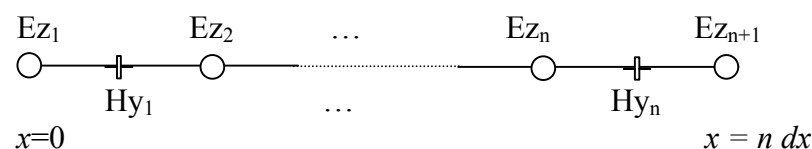

Figure 1: The location and indexing of the discrete variables.

In words: use n cells.
Place Ez at $0 : \Delta x : n\Delta x$ (note, $n + 1$ Ez components in space) and at $t = 0 : \Delta t : ...$
Place Hy at $\Delta x/2 : \Delta x : \Delta x \cdot (n - 1/2)$ (n Hy components in space) and at $t = -\Delta t/2 : \Delta t : ....$
Do NOT restrict your code to the case $\Delta x = 1$.

$$Ez(x,t): Ez_i^k = Ez((i-1)\Delta x, k\Delta t), i = 1,...,n+1, k = 0,1,...$$

$$Hy(x,t): Hy_i^k = Hy((i-1/2)\Delta x, (k-1/2)\Delta t), i = 1,...,n, k = 0,1,...$$
$$(2)$$

$$Ez_i^0 = Ez(t=0) \text{ and } Hy_i^0 = Hy(t = -\Delta t/2) \text{ are initial values, in most cases put to zero.}$$

In vacuum the conductivity is equal to zero ($\sigma = 0$), in non-exotic materials the equivalent magnetic resistivity is zero ($\rho = 0$), the permittivity $\varepsilon \approx 8.8541878 \cdot 10^{-12}$ and the permeability $\mu = 4\pi 10^{-7} \approx 1.256637061 \cdot 10^{-6}$

The grid size Δx can be arbitrarily chosen but the time step Δt is restricted by the maximum CFL-number which in one dimension is one, ($\Delta t < 1 \cdot \Delta x/c$). In this case our recommendation is to use CFL = 1/2. The number of time steps should be chosen such that the wave will propagate around for a while until the whole picture is clear.

Choose a pulse form and a pulse length suitable for the problem at hand. Remember that wave-components with a short wavelength suffer severely from dispersion. The size of the domain n must be large enough to cover the pulse but small enough to get a speedy simulation.

2.1 Basic boundary conditions

Use a Perfect Electric Conductor (PEC) as upper boundary, i.e. $Ez_{n+1}^{k} = 0, k = 0,1,2...$, and a time dependent source function $f(t)$ at the lower boundary, i.e. $Ez_1^{k} = f(k\Delta t), k = 0,1,...$
A suitable function $f(t)$ is e.g. a Gaussian:

$$f(t) = e^{-(t-t_0)^2/t_w^2}, t > 0; = 0, t < 0 \tag{3}$$

where $t_0 = 6t_w$ is a suitable choice. Why? The analytic solution for this boundary condition is $Ez(x,t) = f(t - x/c)$ where $c = 1/\sqrt{\mu\varepsilon}$ is the (phase) speed of propagation.

2.1.1 Assignment 1 (0.5 p)

Make a simple stability test by changing the CFL-number to a few different values.

2.2 Absorbing Boundary Condition (ABC)

Implement Mur 1st order ABC at the upper boundary in the way recommended in *TafloveFDTD*

$$Ez_{n+1}^{k+1} = -Ez_n^{k-1} + a_1(Ez_n^{k+1} + Ez_{n+1}^{k-1}) + a_2(Ez_{n+1}^{k} + Ez_n^{k}),$$
$$a_1 = \frac{cfl-1}{cfl+1}, a_2 = \frac{2}{cfl+1}, cfl = \frac{c\Delta t}{\Delta x} \tag{4}$$

You need initial conditions for Ez_n^{-1} and Ez_{n+1}^{-1}; simply put them to zero.

Note: Equation (4) is a slightly improved formulation compared to the one Mur originally used,

$$Ez_{n+1}^{k+1} = Ez_n^{k} + a_1(Ez_n^{k+1} - Ez_{n+1}^{k}) \tag{5}$$

2.2.1 Assignment 2 (1.0 p)

Make sure that the ABC works correctly. There should be almost no reflections in the ABC. Measure the reflections e.g. by recording e2(t) = L2 norm of Ez as function of time. Plot e2 and the corresponding quantity for the exact solution. Explain what the plot shows.

2.3 Huygens' surface

Implement a Huygens' surface for generation of incident waves. The Huygens' surface divides the domain into two parts. In the left part only the scattered field is calculated and in the right part the total field is calculated.
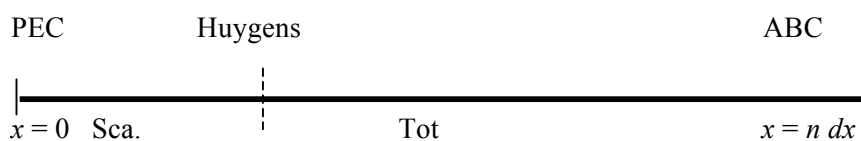

Figure 2: PEC at the lower boundary and ABC at the upper boundary. A Huygens' surface is introduced to generate incident waves.

Introduce the integer parameter *db* which stands for distance-to-boundary. The parameter *db* describes in which cell the Huygens' surface is located. The updating for the magnetic component (total field) immediately to the right of the Huygens' surface is described by

$$Hy_{db}^{k+1} = \frac{1-\alpha}{1+\alpha} \cdot Hy_{db}^{k} + \frac{\beta}{1+\alpha} \cdot (Ez_{db+1}^{k} - Ez_{db}^{k} - Ezinc_{db}^{k})$$

$$\alpha = \frac{\rho \Delta t}{2\mu}, \beta = \frac{\Delta t}{\mu \Delta x} \tag{6}$$

The updating for the electric component (scattered field) immediately to the left of the Huygens' surface is described by

$$Ez_{db}^{k+1} = \frac{1-\gamma}{1+\gamma} \cdot Ez_{db}^{k} + \frac{\delta}{1+\alpha} \cdot (Hy_{db}^{k+1} - Hy_{db-1}^{k+1} - Hyinc_{db}^{k+1})$$

$$\gamma = \frac{\sigma \Delta t}{2\varepsilon}, \delta = \frac{\Delta t}{\varepsilon \Delta x} \tag{7}$$

Note that Hy^{k+1} is known from the latest half time step. If $\sigma = \rho = 0$, $\alpha = \gamma = 0$. Also note that both (6) and (7) are the usual Yee scheme plus a correction term. Hence, the best way to implement (7) is to first perform the Yee update for all electric fields, and then perform

$$Ez_{db}^{k+1} = Ez_{db}^{k+1} - \frac{\delta}{1+\alpha} \cdot Hyinc_{db}^{k+1} \tag{8}$$

and similarly for (6).

The relation between the electric field and the magnetic field for a plane wave is described by the impedance Z

$$|E| = Z \cdot |H|, Z = \sqrt{\frac{\mu}{\varepsilon}} \tag{9}$$

Making sure that these source terms are evaluated with correct *t*-values and taking care of the fact that waves travel with the speed of light gives

$$Hyinc_{db}^{k+1} (= -\frac{1}{Z}f(t - x/c)) = -\frac{1}{Z}f((k+1/2)\Delta t - (db - 1/2)\Delta x/c) \tag{10}$$

$$Ezinc_{db}^{k} (= f(t - x/c)) = f(k\Delta t - (db-1)\Delta x/c)$$

For *f*, you can use the Gaussian in (3). Note that $f(t - x/c)$ (almost) $= 0$ if $t < x/c$.

2.3.1 Assignment 3 (1.0 p)

Make sure that the Huygens' surface generates a right-going wave. The parameter *db* can be put to 3 when all is working fine, but for debugging purposes *db* should be set to a higher value to make sure that (almost) no energy is propagating to the left. However, small amounts of high frequency components will propagate to the left due to discretization errors.
Verify that you have second-order accuracy.

2.3.2 Assignment 4: Dielectric material (1.5 p)

Implement Mur 1st order ABC at the lower boundary and keep the ABC at the upper boundary.

$$Ez_{1}^{k+1} = -Ez_{2}^{k-1} + a_{1}(Ez_{2}^{k+1} + Ez_{1}^{k-1}) + a_{2}(Ez_{1}^{k} + Ez_{2}^{k}) \tag{11}$$

Make the right half a dielectric material with $\varepsilon_r = 4$. Use the Huygens' surface to generate an incident wave. The amplitude of the reflected wave should be R (reflection coefficient) times the amplitude of the incident wave and the transmitted wave should have a amplitude of T (transmission coefficient) times the amplitude of the incident wave where

$$R = \frac{b-1}{b+1}, T = \frac{2}{b+1}, b = \sqrt{\varepsilon_r} (T + R \equiv 1) \tag{12}$$

ABC   Huygens   Dielectric (ε)   ABC

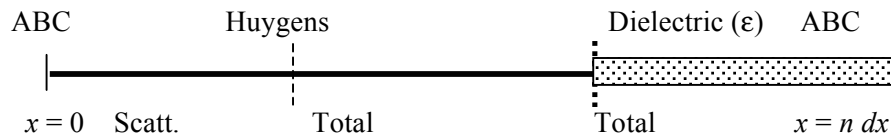$x = 0$ Scatt.   Total   Total $x = n\,dx$

Figure 4: A dielectric material in the right half of the domain.

Remember that the speed of light is used in the formulation of the ABC. This means that the ABC at the upper boundary must be adjusted according the value of εr.

How good agreement do you get with your code compared to the theoretical reflection / transmission coefficients? Also note the wavelength in the dielectric material.

3   Experiments with 2D code fdtd2D.m (Optional 4.0p)

1. The code as is shows how to set material properties when a domain is defined by an expression $F(x,y) <= 0$. Unions and intersections are made by $F <= 0 \,||\, G <= 0$ and $F <= 0 \,\&\, G <= 0$.

2. A persistent source is made by leaving out the Gaussian factor in its definition.

3. Don't try plane waves.

3.1.1 Assignment 5: Familiarize yourself with the code

Run the `fdtd2d` code as is. You will not see much.

a) Change the `n_rec` variable to get 20 steps recorded.

b) Fix the code so it uses a new figure for the movie, set axes (incl. color axis), axis texts, etc. to get an animation to your liking.

Now change the material properties inside the "metal" cylinder to become copper (hint: line 181-214, material number 2.) After this modification there is reflection.

c) The code as is visualizes ex – it seems to be small at the horizontal line through the source. Why? Choose instead a quantity, which has the same magnitude around the wave front.

d) Choose a timestep which is barely *UN*stable and visualize – show every time-step - how the solution blows up. What wavelength does the quickly growing solution have? Why?

e) Finally, remove the reflector again and observe the reflection at the right hand boundary. The code as is computes the L2norm of (ex,ey,hz) and plots it as function of time. Explain why the plot looks the way it does. How efficient are the PML boundaries?

3.1.2 Assignment 6: Virtual optics, focusing by a lens of silicon

Compare results of geometrical optics with computations for finite wavelength (FDTD) on the field emitted from a persistent point source as refracted by a lens bounded by two circular arcs. Check the lens formula from high school,

$$1/f = 1/a + 1/b$$

by moving the point source and recording its image. Use a persistent monochromatic source and believe in the ABC. For the lens, the test should single out the intersection of two circles, i.e.

`((x-a1)^2 + (y-b1)^2 < R1^2) & ((x-a2)^2 + (y-b2)^2 < R2^2)`

Extra: Compute the focal distance *f* also by geometrical optics, given the radii of the arcs and the index of refraction. For a single circular arc it can be done manually by computing the refracted rays by Snell's law, but it is easier to use Matlab, a number of central rays, and to read off their intersection with the optical axis. And for the *two* surfaces of a lens, I think you really want Matlab …

Good Luck!