Homework 4
High-resolution shock-capturing methods
Max. 6.0 p
Deadline Mon April 27, 2015

Topic: High-resolution method for shallow water equations. Sub- and supercritical flow. Limiters and entropy "fix".

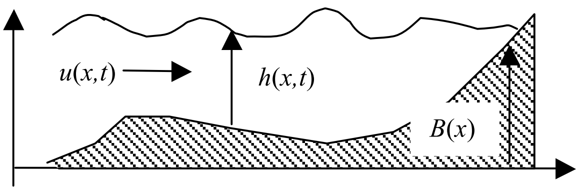
Purpose: Get acquainted with high-resolution methods for non-linear system of hyperbolic equations. Be able to handle different boundary conditions given different flow conditions.

Instructions: Write a report with the plots and answers to the questions posed. Make sure the plots are annotated and there is explanation for what they illustrate. Append the code to the report.

1 Shallow water with non-horizontal bottom (0.25 p)

The shallow water model of HW2 is now extended to a non-horizontal bottom "bathymetry" $B(x)$,

$$\begin{pmatrix} h_t \\ hu_t \end{pmatrix} + \begin{pmatrix} h_x u + h u_x \\ huu_x + gh(h_x + B_x) \end{pmatrix} = 0$$



a) Show that still water ($u = 0$) must have, as it should, a horizontal water level.
b) Write the equation in conservation form for h and $m = hu$:

$$\begin{pmatrix} h \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ f_2(h,m) \end{pmatrix}_x = \begin{pmatrix} 0 \\ s(h,m,x) \end{pmatrix}$$

It is important that the source function s should not contain derivatives of h or m.

In the remaining part of the homework your job is to write a Roe-solver and extend it to a high-resolution scheme for the model.

2 First order Roe scheme

The first order Roe scheme has the numerical flux

$$\mathbf{F}^n_{i+1/2} = \frac{1}{2}(\mathbf{f}(\mathbf{Q}^n_i) + \mathbf{f}(\mathbf{Q}^n_{i+1})) - \frac{1}{2}\left|\hat{\mathbf{A}}_{i+1/2}\right|(\mathbf{Q}^n_{i+1} - \mathbf{Q}^n_i),$$

where $\hat{\mathbf{A}}_{i+1/2} = \hat{\mathbf{A}}(\mathbf{Q}_i, \mathbf{Q}_{i+1})$ is the Roe-average matrix at the i+1/2 interface (Leveque 15.3.3). When you implement the scheme it may be worthwhile to use the form in Leveque eq. (15.51) instead,

$$\mathbf{F}^n_{i+1/2} = \frac{1}{2}(\mathbf{f}(\mathbf{Q}^n_i) + \mathbf{f}(\mathbf{Q}^n_{i+1})) - \frac{1}{2}\sum_{p=1}^{2}\left|\hat{\lambda}^p_{i+1/2}\right| W^p_{i+1/2}$$

which is easier to extend to the high-resolution scheme later on. You need eigenvalues and eigenvectors to the Roe matrix to compute the waves $W^p_{i+1/2}$. Formulas can be found in Leveque 15.3.3.

2.1 Roe tests with flat bottom (1.0 p)

Implement the first order Roe scheme for the case of a flat bottom, i.e. $s(h,m,x) = 0$. Let the channel be L long, with n cells $\Delta x = L/n$. The bottom is horizontal ($B = 0$) and the water depth is H. Use "wall" boundary conditions at $x = 0$ and L, i.e. $h_0 = h_1$, $m_0 = -m_1$, (and similarly at $x = L$). Choose initial data to give a right running Gaussian pulse with height a and width w starting at $x = L/2$,

$$h(x,0) = H + ae^{-(x-L/2)^2/w^2}$$

Take $H = 1$, $w = 0.1L$, $a = H/5$ or so, to make the wave almost linear.

a) Explain how to choose $m(x,0)$ to make a *single* pulse (not two!) based on the linearized problem.
b) Use this initial data and run the Roe solver with n = 80, 160, 320 and the CFL number as large as possible without instability, until the wave has been fully reflected at L. Plot the wave shapes.
c) Compare with solutions obtained using your Lax-Friedrichs solver from HW2.
d) Comment about order of accuracy, dissipation and dispersion.
e) Then try also a higher pulse, say $a = 2H$. Comment on how well the single pulse initial data works.

2.2 Steady solutions with non-horizontal bottom

Now consider the case with a bump on the bottom, say

$$B(x) = \begin{cases} B_0 \cos^2\left(\dfrac{\pi(x-L/2)}{2r}\right), & |x - L/2| < r, \\ 0, & |x - L/2| \ge r. \end{cases}$$

Variable B introduces a source term, so a conservative first order scheme is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\hat{F}_{i+1/2}^n - \hat{F}_{i-1/2}^n) + \Delta t \cdot S(Q_i^n)$$

where the $\hat{F}_{i+1/2}^n$ are the numerical flux functions for $S = 0$.

2.2.1 Boundary conditions (0.25 p)

Since the problem is non-linear, the number of boundary conditions that should be prescribed at each boundary depends on the solution itself.

The flow is called *sub-critical* if the water velocity is less than the characteristic speed, i.e. if $|u(x)| < \sqrt{gh(x)}$.

The flow is called *super-critical* if the water velocity is greater than the characteristic speed, i.e. if $|u(x)| > \sqrt{gh(x)}$.

a) Suppose we want to use Dirichlet boundary conditions for $u(x)$ and/or $h(x)$. Determine where we should put them when the flow is sub- and super-critical respectively in a neighborhood of the two boundaries $x = 0$ and $x = L$.

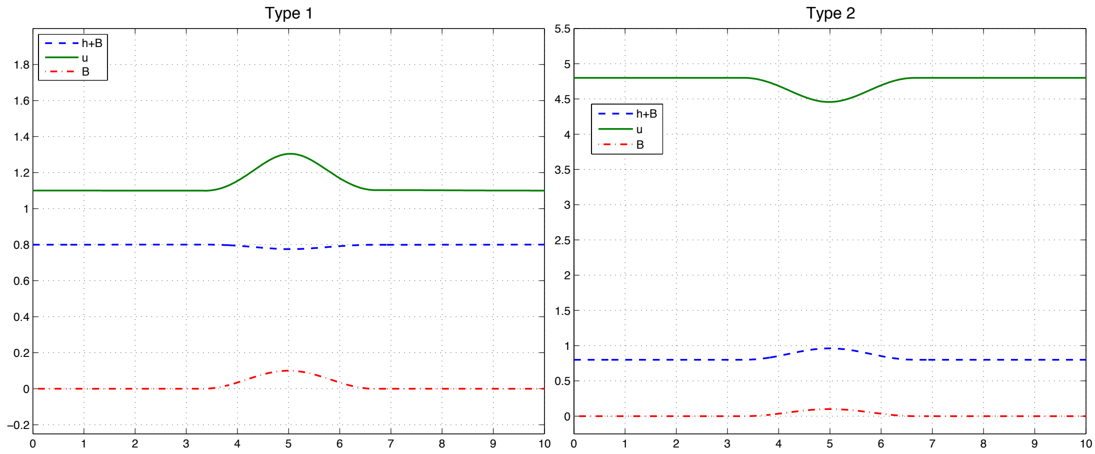
2.2.2 Smooth steady state solutions (0.75 p)

Now **compute steady solutions** for the non-horizontal bottom by running a time-accurate simulation for a LONG time. $x = 0$ is an inflow boundary and $x = L$ is an outflow boundary.

For the bump, use $B_0 = H/10$, $r = L/6$. Implement inflow by prescribing the value in the ghost cell and outflow by extrapolation.

Find initial data and boundary conditions at $x=0$ and $x=L$ which give the following types of solutions (cf. figures below):

a) All sub-critical flow: water accelerates on the bump (but remains sub-critical), and then slows down again. Show a plot of the solution at steady state.
b) All super-critical flow: water decelerates on the bump (but remains super-critical), then accelerates again to super-critical outflow. Show a plot of the solution at steady state.



2.2.3 Steady states with transonic rarefaction (1.0 p)

You shall now find steady solutions where **part of the solution is sub-critical and part of it is super-critical.** The two parts will be connected by shocks and rarefactions, which will then be transonic. These rarefactions will not be captured correctly with the Roe scheme. To get it right one must add an entropy fix.

Use the same setup as in the previous exercise but now **find initial data and boundary conditions** at $x=0$ and $x=L$ which give the following types of steady solutions:

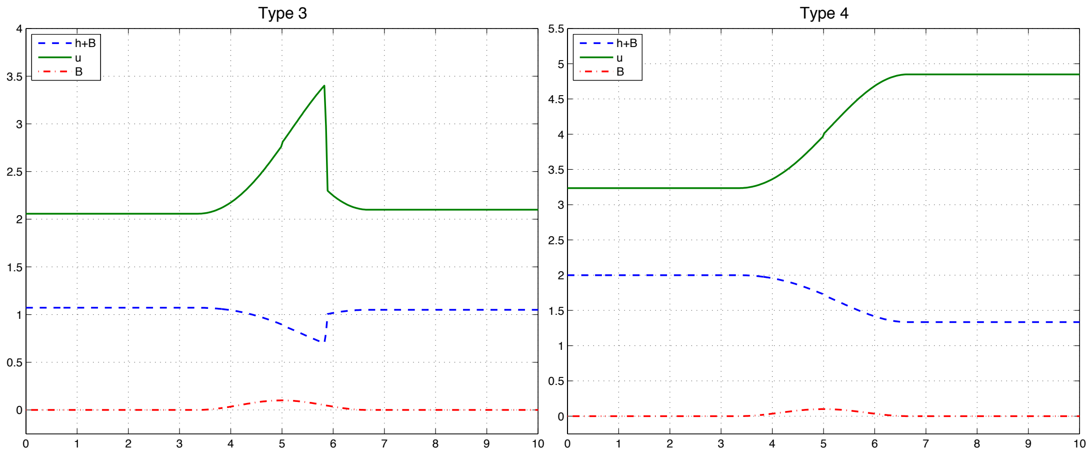
a) Sub-critical inflow, which accelerates to super-critical on the bump and then, reverts to sub-critical through a shock. Show a plot of the solution at steady state.
b) Sub-critical inflow, which accelerates to super-critical on the bump and stays super-critical. Show a plot of the solution at steady state.

See figures below for examples. In the type 3, we can see a small entropy violating shock. The reason for this is that the viscous term in the Roe flux is too small in the case of a transonic rarefaction wave.

c) Try solving the same problem with Lax-Friedrichs. Show plots that demonstrate the difference in dissipation between the two schemes. How many grid points would you need to get a reasonable looking solution?

Optional (0.75):

d) To get rid of the entropy violating shock, implement Harten's entropy fix (Leveque p 326) and compute the steady solutions. Indicate in your plots where the solutions are sub- and super-critical. Try varying the parameter δ. How should it scale with Δx?



Note the small entropy-violating "shocklet" in type 3… the entropy fix should be strengthened.

3 High-resolution Roe scheme

Extend to a Roe high-resolution scheme as explained in Leveque 6.13-6.15, 15.4. You need to add a correction $\tilde{\mathbf{F}}^n_{i+1/2}$ to the flux, $\mathbf{F}^n_{i+1/2}$, (see eq. (15.63) in Leveque) where

$$\tilde{\mathbf{F}}^n_{i+1/2} = \frac{1}{2}\sum_{p=1}^{2}\left|\hat{\lambda}^p_{i+1/2}\right|\left(1-\frac{\Delta t}{\Delta x}\left|\hat{\lambda}^p_{i+1/2}\right|\right)\tilde{W}^p_{i+1/2}$$

and $\tilde{W}^p_{i+1/2}$ is a limited version of $W^p_{i+1/2}$. Use the *minmod* limiter and note the comments (Leveque pp. 121-122) on how to compare the waves from a cell interface to its upstream neighbor: take scalar products and define $\theta^n_{i+1/2}$ this way. It is sufficient to apply the limiter to the waves in the interior. Just use $W^p_{i+1/2}$ itself at the boundary. A vectorized `matlab` implementation is preferable, but use loops if you like.

3.1.1 High-resolution tests (2.0 p)

a) Run the flat bottom tests in Section 2.1. Check that the dissipation is much smaller than for the Lax-Friedrichs and the standard Roe scheme. Show plots comparing the solutions after short and long time.
b) Check that smooth extrema (no shocks) get clipped – the effect of the limiter. Show a zoomed plot!
c) Try the four different types of solution above (see 2.2.2 and 2.2.3). Compare with the standard Roe scheme with the same number of grid points.