Homework 6

Iterative Methods

due March 2, 2015

A common method for the computation of time-dependent incompressible flows is the so-called *projection method*, which will be discussed in detail during the course. In this method, the main effort at each time step is to solve a Poisson equation for the pressure. When discretising the Navier–Stokes equations on a staggered grid, one can show that homogeneous Neumann boundary conditions for the pressure should be imposed.

Let us consider a two-dimensional flow. Then we have

$$\Delta p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = f, \quad \frac{\partial p}{\partial n} = 0 \text{ on boundaries.} \tag{1}$$

Since the boundary condition on all boundaries involves only derivatives of the pressure, to find an unique solution, we need to specify the value of pressure at some point in the domain.

In this homework you are asked to apply an iterative method to solve the equation above for a domain $0 \le x \le L_x$ and $0 \le y \le L_y$ (set $L_x = L_y = 1$ for this problem). The forcing function f is explicitly given as

$$f(x, y) = \cos(\pi x)\cos(\pi y) \ . \tag{2}$$

Equation (1) should be discretised with second-order central differences. Use a grid with $N_x \times N_y$ cells with spacing $h_x = L_x/N_x$ and $h_y = L_y/N_y$, see figure 1. Assemble the numerical solution $\underline{p}$ and the inhomogeneous part $\underline{f}$ as one-dimensional vectors of the form

$$\underline{p} = (p_{1,1}, p_{2,1}, ..., p_{N_x,1}, p_{1,2}, ..., p_{N_x-1,N_y}, p_{N_x,N_y})^T \ , \tag{3}$$

$$\underline{f} = (f_{1,1}, f_{2,1}, ..., f_{N_x,1}, f_{1,2}, ..., f_{N_x-1,N_y}, f_{N_x,N_y})^T \ , \tag{4}$$

where $p_{i,j}$ and $f_{i,j}$ are the values of p and f on the cell centres with the coordinates $x_i = (i - 1/2)h_x$ and $y_j = (j - 1/2)h_y$ (the black dots in figure 1).

Then, equation (1) is rewritten as a linear system of $N_x \times N_y$ equations of the form $\underline{\underline{A}}\ \underline{p} = \underline{f}$, in which the boundary conditions have yet to be implemented. The homogeneous Neumann condition means that the pressure in cells close to boundary should be equal to those at dummy nodes (gray dots in figure 1), i.e.,

$$p_{1,j} = p_{0,j}, \quad p_{N_x,j} = p_{N_x+1,j}, \quad p_{i,1} = p_{i,0}, \quad p_{i,N_y} = p_{i,N_y+1} \ . \tag{5}$$

This corresponds to a discretised homogeneous Neumann condition of order one.

Two iterative methods will be used in this problem: Gauss-Seidel (GS) and Successive Over-Relaxation (SOR). Both of these methods, when applied to the discretised Poisson equation using central differences, can be written explicitly as

$$p_{i,j}^{(m+1)} = (1-\omega)p_{i,j}^{(m)} + \frac{\omega}{2(1+\beta^2)}\left[p_{i+1,j}^{(m)} + p_{i-1,j}^{(m+1)} + \beta^2(p_{i,j+1}^{(m)} + p_{i,j-1}^{(m+1)}) - h_x^2 f_{i,j}\right] \ , \tag{6}$$

where the Gauss-Seidel method corresponds to the case of $\omega = 1$. $p^{(m)}$ denotes the iterative solution after m number of iterative steps and $\beta = h_x/h_y$.

Applying SOR to the discrete Laplace operator, including boundary conditions (5), along $i = 1$ yields

$$
\begin{aligned}
p_{1,1}^{(m+1)} &= (1-\omega)p_{1,1}^{(m)} + \frac{\omega}{1+\beta^2}\left[p_{2,1}^{(m)} + \beta^2 p_{1,2}^{(m)} - h_x^2 f_{1,1}\right] \ , \\
p_{1,j}^{(m+1)} &= (1-\omega)p_{1,j}^{(m)} + \frac{\omega}{2(1+\beta^2)-1}\left[p_{2,j}^{(m)} + \beta^2 p_{1,j+1}^{(m)} + \beta^2 p_{1,j-1}^{(m+1)} - h_x^2 f_{1,j}\right] (j = 2, N_y - 1) \ , \\
p_{1,N_y}^{(m+1)} &= (1-\omega)p_{1,N_y}^{(m)} + \frac{\omega}{1+\beta^2}\left[p_{2,N_y}^{(m)} + \beta^2 p_{1,N_y-1}^{(m+1)} - h_x^2 f_{1,N_y}\right] \ .
\end{aligned}
$$

Note that SOR is applied after the inclusion of the boundary conditions. The equations along $i = N_x$, $j = 1$ and $j = N_y$ should be modified in a similar way. As mentioned before, since the boundary conditions are only on the derivatives of p, the value of the pressure will only be determined up to an additive constant. To fix this constant replace the equation at node $(1, 1)$ with $p_{1,1} = 0$.

Your task is to

a) Investigate if the problem is well posed. Consider in particular the boundary conditions (Compatibility condition!).

b) Choose $N_x = N_y = 30$. First find the numerical solution $\underline{p}_{\text{direct}}$ of the problem by computing the direct solution of the 900×900 matrix. Present the solution in a suitable plot (contour or 3D).

c) Implement the GS and the SOR methods using the relaxation factor $\omega = 1.5$. Choose as initial guess for the inner points $\underline{p}^{(0)} = \underline{0}$. The iterative solution after m iterations is denoted $\underline{p}^{(m)}$. Show the convergence by plotting the residual

$$R^{(m)} = ||\underline{\underline{A}}\ \underline{p}^{(m)} - \underline{f}||_2 = \sqrt{(\underline{\underline{A}}\ \underline{p}^{(m)} - \underline{f})^T \cdot (\underline{\underline{A}}\ \underline{p}^{(m)} - \underline{f})} \tag{7}$$

versus the number of iterations m in a semi-logarithmic plot.

d) Plot the 2-norm of the difference between the GS and SOR methods from part c) with the direct solution from part b),

$$\varepsilon^{(m)} = ||\underline{p}^{(m)} - \underline{p}_{\text{direct}}||_2 \ , \tag{8}$$

with respect to m in a semi-logarithmic plot. Compare and discuss the various iterative methods, and compare with the residuals obtained in part c). Can you improve SOR using a different ω?

e) Repeat part d) with $N_x = N_y = 20$. Compare the results for $\varepsilon^{(m)}$ versus m for $N_x = N_y = 20$ and $N_x = N_y = 30$. How does the convergence rate change with respect to $N = N_x = N_y$? Estimate the convergence rate from the slope of the residual plot.

Hint: A template of the program can be found on the homepage.

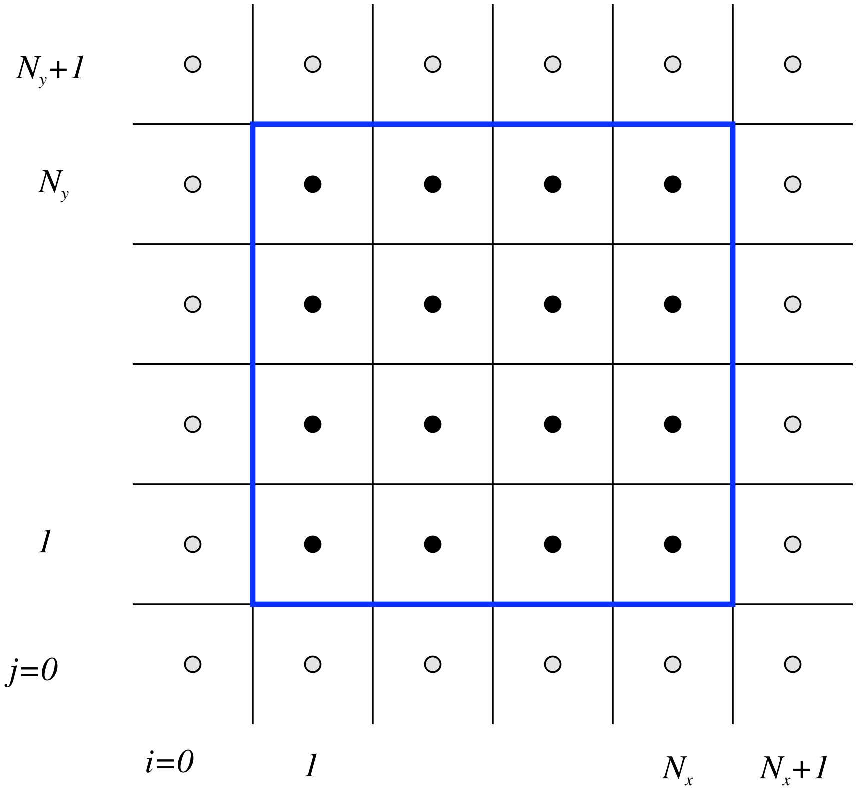

Figure 1: Sketch of the grid. Black dots denote the $(N_x \times N_y)$ positions in which the pressure is evaluated, whereas gray dots indicate dummy cells situated outside the computational domain.