We consider r.v.s

$X_1, \dots, X_V$

Scope of factor —

its r.v.s.

Given:

- UGM $(G, \{\psi^i\}_{i \in [n]})$
- Junction tree (T, B) of G of width w.

$\forall \psi^i \; \exists \, t \in V(T)$ with $B(t)$ "containing" scope of ψ^i

We write $\varphi^i(\bar{x})$ even if φ^i has smaller scope than $\bar{x}$.

We want to compute "the joint"

$$\varphi(\bar{x}) = \prod_i \varphi^i(\bar{x}).$$

Evidence can be incorporated into factors as follows:

$$\varphi^i(X_{v\setminus e}) := \varphi^i(X_{v\setminus e}, X_e)$$

So w.l.o.g. assume no evidence.
Root T in arb. vertex r. Assign each ψ^i to unique t s/t $B(t)$ contains its scope.
Let

$$\psi_t(X_{B(t)}) := \prod_{\substack{\psi^i \text{ ass.} \\ \text{to } t}} \psi^i(\bar{X}_{B(t)})$$

Notice

$$\psi(\bar{X}) = \prod_{t \in V(T)} \psi_t(X_{B(t)})$$

Subproblems and subsolutions

$$(1)\quad \psi_t^r(x_{B(t)}) := \sum_{x_{U(t)}} \prod_{s \in V(T_t)} \psi_s(x_{U(t)}, x_{B(t)})$$

where $U(t) := B(V(T_t)) \setminus B(t)$

(remember $B(W) = \bigcup_{t \in W} B(t)$

for $W \subseteq V(T)$)

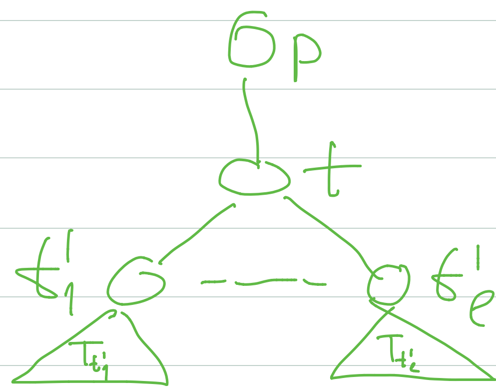
and

(2) For a child t' of t

$$\psi^r_{t'\to t} := \sum_{w(t')} \prod_{s\in V(T_{t'})} \psi_s(x_{w(t')}, X_{B(t)})$$

Where $W(t') := V(T_{t'}) \setminus B(t)$

Consider configuration:



Notice

$$\psi^r_t(X_{B(t)}) = \sum_{x_{U(t)}} \prod_{s\in V(T_t)} \psi_s(x_{U(t)}, X_{B(t)})$$

$$= \psi_t(X_{B(t)}) \sum_{x_{U(t)}} \prod_{j=1}^{\ell} \prod_{s\in V(T_{t'_j})} \psi_s(x_{U(t)}, X_{B(t)})$$

[Since $(B(V(T_{t'_j})) \cap B(t)) = (B(V(T_{t'_{j'}})) \cap B(t)) = \emptyset \ \forall j \neq j'$]

$$= \psi_t(X_{B(t)}) \prod_{j=1}^{\ell} \sum_{x_{W(t'_j)}} \prod_{s\in V(T_{t'_j})} \psi_s(x_{W(t'_j)}, X_{B(t)})$$

$$= \psi_t(X_{B(t)}) \prod_{j=1}^{\ell} \psi^r_{t_j' \to t}(X_{B(t)})$$

This shows how to compute ψ^r_t (given all comp. for its proper descendants).

Notice

$$\psi^r_{t \to p}(X_{B(p)}) = \sum_{X_{W(t)}} \prod_{s \in V(T_t)} \psi_s(x_{W(t)}, X_{B(p)})$$

$$= \sum_{X_{B(t) \setminus B(p)}} \psi^r_t(x_{B(t) \setminus B(p)}, X_{B(p)})$$

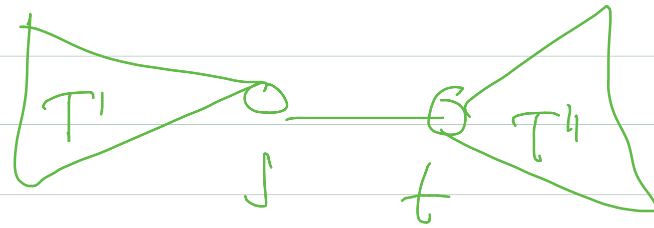
This shows how to compute $\psi^r_{t \to p}$ (given all other comp. for its desc.).

Finally we can sum out $X_{B(r)}$ from $\psi^r_r(X_{B(r)})$.

Notice

- We can root anywhere

— For conf.



any root r' in T' gives the same $\psi^{r'}_{t \to s}$ and

any root r'' in T'' gives the same $\psi^{r''}_{t \to s}$

Conclusion

(1) only 2 possible messages for each edge

(2) all "bag marginals" $\psi(X_{B(t)})$ can be comp. in time $O(|E(T)|\, c^w) = O(|V(G)|\, c^w)$ for some const. c (which dep. on # of values in our categorical CPDs)