Complex Gaussian, Continuous Time Stochastic Processes

Course: Foundations in Digital Communications

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6th lecture

Recapitulation

What did we do last lecture?

Outline - Motivation

- Passband processing is done in complex domain and Gaussian is the most important distribution. Therefore, let's have a look at these distributions:
  - Complex Gaussian and Circular Symmetry (chap 24)
- Most physical signals realizations are continuous. Thus an extension of the notions and a solid understanding of stochastic processes is required.
  - Continuous Time Stochastic Processes (chap 25)

Standard Complex Gaussian

Standard Complex Gaussian

Complex RV W whose real and imaginary part are independent $\mathcal{N}(0, 1/2)$ RVs.

$$f_W(w) = \frac{1}{\pi} e^{-|w|^2}, \quad w \in \mathbb{C}$$

$\mathbb{E}[W] = 0$ and $\mathrm{Var}[W] = \mathbb{E}\left[|W|^2\right] = 1$

Standard complex Gaussian RV are

- **proper** since $\mathbb{E}[W] = 0$ and $\mathbb{E}\left[W^2\right] = 0$
- **radially-symmetric** since $f_W(w)$ depends on $|w|$ only

Circular Symmetry

Circular Symmetry

CRV Z is *circularly-symmetric* if for any $\phi \in [-\pi, \pi)$ we have

$$e^{-i\phi} Z \overset{\mathcal{L}}{=} Z$$

If CRV Z has a density, then we have the following equivalences:

- Z is circularly-symmetric
- Z has a radially-symmetric density function
- Z can be written as $Z = Re^{i\Theta}$, RV R and Θ with $R \perp \Theta$, $R \in [0, \infty)$, $\Theta \sim \mathcal{U}([-\pi, \pi))$
- **Example:** CRV $Z = e^{i\Phi}$, $\Phi \sim \mathcal{U}([-\pi, \pi))$ is circularly-symmetric, but does not have a density.

Properness and Circular Symmetry

- Circular symmetry considers whole distribution, properness considers first two moments only.

1. Every finite-variance circularly-symmetric CRV is proper.
2. Not every proper CRV is circularly-symmetric

Proof:

$$\mathbb{E}\left[Z^k\right] = \mathrm{e}^{-ik\Phi}\mathbb{E}\left[(\mathrm{e}^{i\Phi}Z)^k\right] = \mathrm{e}^{-ik\Phi}\mathbb{E}\left[Z^k\right], \quad \Rightarrow \; \mathbb{E}\left[Z^k\right] = 0, k = 1, 2$$

- Counterexample: CRV Z with $1+i, 1-i, -1+i, -1-i$ equiprobable is proper, but $\mathrm{e}^{i\pi/4}$ takes values $\sqrt{2}, -\sqrt{2}, \sqrt{2}i, -\sqrt{2}i$, thus not same distribution. □
- Result also holds for the random vectors.

Complex Gaussian

Complex Gaussian

A CRV is complex Gaussian if real and imaginary part are jointly real Gaussian. It is centered if the mean is zero.

By some algebraic comparisons we can see

W standard complex Gaussian. For a centered complex Gaussian Z

1. exists $\alpha, \beta \in \mathbb{C}$ such that $Z \stackrel{\mathscr{L}}{=} \alpha W + \beta W^*$
2. is proper iff exists $\alpha \in \mathbb{C}$ such that $Z \stackrel{\mathscr{L}}{=} \alpha W$

- For Z proper complex Gaussian from (ii) it follows

  $$f_Z(z) = \frac{f_W(z/\alpha)}{|\alpha|^2} = \frac{1}{2|\alpha|^2} e^{-\frac{|z|^2}{|\alpha|^2}}, z \in \mathbb{C}$$

  $\Rightarrow$ radially-symmetric, which implies circularly-symmetric.
- **A complex Gaussian is circular-symmetric iff it is proper!**

Complex Gaussian Vectors

- **Standard complex Gaussian vector:** Components are iid standard complex Gaussian:

$$f_W(w) = \frac{1}{\pi^n} e^{-w^\dagger w}, \quad w \in \mathbb{C}^n.$$

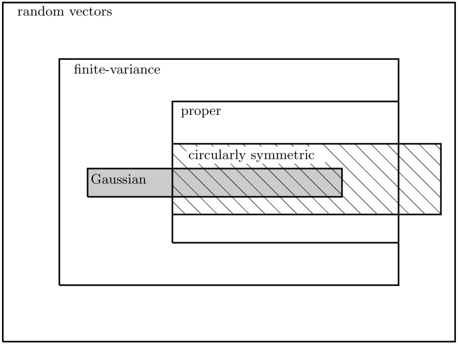
- **Circularly-Symmetric**: $e^{-i\phi} Z \overset{\mathscr{L}}{=} Z$ for every $\phi \in [-\pi, \pi)$
  $\Leftrightarrow$ for every $\alpha \in \mathbb{C}^n$, the CRV $\alpha^T Z$ is circularly-symmetric.
- Z circularly-symmetric $\Rightarrow$ AZ circularly-symmetric, $A \in \mathbb{C}^{n \times m}$
- $\mathbb{E}\left[ZZ^\dagger\right]$ and $\mathbb{E}\left[ZZ^T\right]$ specify a centered complex Gaussian vector Z
  $\Rightarrow$ $\mathbb{E}\left[ZZ^\dagger\right]$ specifies a *proper* complex Gaussian vector
- **Complex Gaussian Vector:** $\exists A, B \in \mathbb{C}^{n \times m}$ and $\mu \in \mathbb{C}^n$ such that $Z \overset{\mathscr{L}}{=} AW + BW^* + \mu$, $W \in \mathbb{C}^m$ standard complex Gaussian
- Complex Gaussian vector is proper iff it is circularly-symmetric

Proper Complex Gaussian Vectors

- Pdf of **proper** complex Gaussian vector Z with $K \succ \mathbf{0}$

$$f_Z(z) = \frac{1}{\pi^n \det K} e^{-z^\dagger K^{-1} z}, \quad z \in \mathbb{C}^n$$

- Relationship between complex random vectors:



Let's take a break!

Continuous-Time Stochastic Processes (SP)

- SP $(X(t), t \in \mathbb{R})$ defined on common probability space $(\Omega, \mathcal{F}, P)$

$$\boldsymbol{X} : \Omega \times \mathbb{R} \to \mathbb{R}, \qquad (\omega, t) \mapsto X(\omega, t)$$

  - Random variable: $\omega \mapsto X(\omega, t)$
  - Function of time: $t \mapsto X(\omega, t)$ (trajectory)
- SP $(X(t))$ is of zero mean (finite variance) if RV $X(t)$ is centered (finite variance) for every $t \in \mathbb{R}$

Finite dimensional distributions (FDD) of a SP

Family of joint distributions $(X(t_1), \ldots, X(t_n))$ for any positive integer n and epochs $t_1, \ldots, t_n \in \mathbb{R}$

- $(X(t))$ and $(Y(t))$ are **independent SP** if $(X(t_1), \ldots, X(t_n))$ and $(Y(t_1), \ldots, Y(t_n))$ are independent $\forall n$ and $t_1, \ldots t_n \in \mathbb{R}$.
- CDF: $F_n(\xi_1, \ldots, \xi_n; t_1, \ldots, t_n) \triangleq \mathbb{P}\left[X(t_1) \leq \xi_1, \ldots, X(t_n) \leq \xi_n\right]$

Kolmogorov's Existence Theorem

Kolmogorov's Existence Theorem

Let $(G_n)_n$ sequence of functions $G_n : \mathbb{R}^n \times \mathbb{R}^n \to [0,1]$ satisfying

- $G_n(\cdot; t_1, \ldots, t_n)$ is a valid joint distribution $\forall n \geq 1$ and $t_1, \ldots, t_n \in \mathbb{R}$
- **symmetry property** holds for all n, t_k, ξ_k and permutations π $G_n(\xi_{\pi(1)}, \ldots, \xi_{\pi(n)}; t_{\pi(1)}, \ldots, t_{\pi(n)}) = G_n(\xi_1, \ldots, \xi_n; t_1, \ldots, t_n)$
- and **consistency property** for all t_k, ξ_k $\lim_{\xi_n \to \infty} G_n(\xi_1, \ldots, \xi_n; t_1, \ldots, t_n) = G_{n-1}(\xi_1, \ldots, \xi_{n-1}; t_1, \ldots, t_{n-1})$

then $\exists$ SP $(X(t))$ whose FDDs are given by $\{G_n(\cdot;\cdot)\}$ in the sense

$$\mathbb{P}\left[X(t_1) \leq \xi_1, \ldots, X(t_n) \leq \xi_n\right] = G_n(\xi_1, \ldots, \xi_n; t_1, \ldots, t_n)$$

- Symmetry and consistency are sufficient for existence of FDDs of some SP

Gaussian Stochastic Process (SP)

Gaussian stochastic process

SP $(X(t))$ is a **Gaussian stochastic process** if $(X(t_1), \ldots, X(t_n))^T$ is a Gaussian random vector for every n.

One reason why Gaussian SP are tractable:

FDDs for *centered* Gaussian SP: All its FDDs are determined by autocovariance function $(t_1, t_2) \mapsto \mathrm{Cov}\,[X(t_1), X(t_2)]$

Proof: Since Gaussian SP is centered, mean is zero and covariance matrix for any n-vector is given by $\left(\mathrm{Cov}\left[X(t_j), X(t_k)\right]\right)_{1 \le j,k \le n}$. $\square$

- $\tau \mapsto \mathrm{Cov}\,[X(t), X(t+\tau)]$ if wide-sense stationary Gaussian SP

Stationary Continuous-Time Process

Stationary

A SP is **stationary** (aka strict sense stationary, strongly stationary) if

$$\big(X(t_1+\tau),\ldots,X(t_n+\tau)\big) \overset{\mathcal{L}}{=} \big(X(t_1),\ldots,X(t_n)\big)$$

Wide-Sense Stationary

A SP is **wide-sense stationary** (aka second-order stationary, weakly stationary) if the following are met

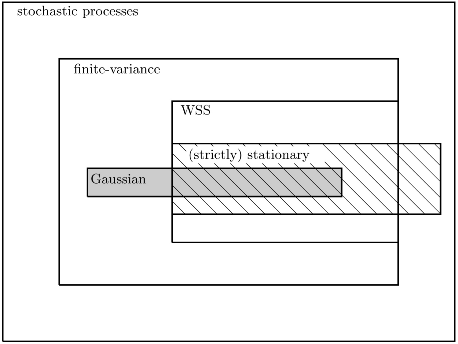
1. it is of finite variance,
2. $\mathbb{E}\left[X(t)\right] = \mathbb{E}\left[X(t+\tau)\right]$ for all $t,\tau \in \mathbb{R}$,
3. covariance between samples

$$\mathrm{Cov}\left[X(t_1)X(t_2)\right] = \mathrm{Cov}\left[X(t_1+\tau)X(t_2+\tau)\right] \quad \forall t_1,t_2,\tau \in \mathbb{R}.$$

- Autocovariance fct of WSS SP $K_{XX}(\tau) \triangleq \mathrm{Cov}\left[X(t+\tau)X(t)\right]$

Stationary Gaussian Stochastic Process

- Stationary Gaussian Stochastic Processes
  - A Gaussian SP is stationary iff it is WSS.
  - FDDs of a centered stationary Gaussian SP are fully specified by its autocovariance function.
- Relationship between stochastic processes



Properties Autocovariance Function

- If the autocovariance fct of WSS SP is continuous at the origin, then it is a uniformly continuous function.
- The autocovariance function $K_{XX}(\tau)$ of a WSS SP $X(t)$ is a
  - symmetric function: $K_{XX}(\tau) = K_{XX}(-\tau)$
  - positive definite function: $\sum_{\nu=1}^{n} \sum_{\nu'=1}^{n} \alpha_\nu \alpha_{\nu'} K_{XX}(t_\nu - t_{\nu'}) \geq 0$
- *Proof:* 5-minute exercise.

Every symmetric positive definite function is the autocovariance function of some stationary Gaussian SP.

Proof: Construct covariance matrix of centered Gaussian vector and then use Kolmogorov's Existence Theorem. $\square$

PSD of a Continuous-Time SP

PSD

WSS SP $(X(t))$ is of PSD S_{XX} if S_{XX} is a non-negative, symmetric, integrable function with

$$K_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f) e^{i2\pi f\tau} \mathrm{d}f, \quad \tau \in \mathbb{R}.$$

$$\mathrm{Var}\left[X(t)\right] = K_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(f) \, \mathrm{d}f$$

- Every non-negative, symmetric, integrable fct is the PSD of some stationary Gaussian SP whose autocovariance function is continuous.
- A **more general concept:** Not every WSS SP with a continuous acf has a PSD, but it has a *spectral distribution function*.

Average Function

- A SP is *measurable* if the mapping $(\omega, t) \mapsto X(\omega, t)$ is a measurable mapping.

Power in Centered WSS SP

$(X(t))$ a measurable, centered, WSS SP with acf K_{XX}, then the RV

$$\omega \mapsto \frac{1}{b-a} \int_a^b X^2(\omega, t) \mathrm{d}t$$

satisfies $\frac{1}{b-a} \mathbb{E}\left[\int_a^b X^2(\omega, t) \mathrm{d}t\right] = K_{XX}(0)$, which denotes the power.

Proof:

$$\mathbb{E}\left[\int_a^b X^2(\omega, t) \mathrm{d}t\right] = \int_a^b \mathbb{E}\left[X^2(\omega, t)\right] \mathrm{d}t = \int_a^b K_{XX}(0) \mathrm{d}t = (b-a) K_{XX}(0)$$

□

Integral of a SP – Mean Value

- For detecting and processing of continuous-time signals we are interested in stochastic integrals.

$$\omega \mapsto \int_{-\infty}^{\infty} X(\omega, t) s(t) \mathrm{d}t$$

$s : \mathbb{R} \to \mathbb{R}$ integrable function, $(X(t))$ measurable WSS SP

- We derive heuristically mean and variance for centered WSS SP, the extension to non-centered $X(\omega, t) + \mu$ is straightforward.
  - Mathematically, some issues have to be resolved: $t \mapsto X(\omega, t)s(t)$ has to be integrable for almost all ω. The result of integration has to be a RV (check book and references).
- **Mean:**

$$\mathbb{E}\left[\int_{-\infty}^{\infty} X(\omega, t) s(t) \mathrm{d}t\right] = \int_{-\infty}^{\infty} \mathbb{E}\left[X(\omega, t)\right] s(t) \mathrm{d}t = \mathbb{E}\left[X(0)\right] \int_{-\infty}^{\infty} s(t) \mathrm{d}t$$

Integral of a SP – Variance

- Variance:

$$\mathrm{Var}\left[\int_{-\infty}^{\infty} X(\omega,t)s(t)\mathrm{d}t\right] = \mathbb{E}\left[\left(\int_{-\infty}^{\infty} X(\omega,t)s(t)\mathrm{d}t\right)^2\right]$$

$$= \mathbb{E}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\omega,t)s(t)X(\omega,\tau)s(\tau)\mathrm{d}t\mathrm{d}\tau\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(t)\underbrace{\mathbb{E}\left[X(\omega,t)X(\omega,\tau)\right]}_{=K_{XX}(t-\tau)}s(\tau)\mathrm{d}t\mathrm{d}\tau$$

$$= \int_{-\infty}^{\infty} K_{XX}(\sigma)\underbrace{\int_{-\infty}^{\infty} s(\sigma+\tau)s(\tau)\mathrm{d}\tau}_{=R_{ss}(\sigma)}\mathrm{d}\sigma = \int_{-\infty}^{\infty} K_{XX}(\sigma)R_{ss}(\sigma)\mathrm{d}\sigma$$

- Frequency domain: $\mathrm{Var}\left[\int_{-\infty}^{\infty} X(\omega,t)s(t)\mathrm{d}t\right] = \int_{-\infty}^{\infty} S_{XX}(f)|\hat{s}(f)|^2\mathrm{d}f$

Linear Functionals of Stationary Gaussian Processes

- The extension of the argument that the linear combination of Gaussian RV is a Gaussian RV extends for the linear combination of Gaussian WSS SP (very valuable result)!

Additionally to the previous let $(X(t))$ be **Gaussian SP**, $\alpha_\nu \in \mathbb{R}$, then

$$\omega \mapsto \int_{-\infty}^{\infty} X(\omega, t)s(t)\mathrm{d}t + \sum_{\nu=1}^{n} \alpha_\nu X(\omega, t_\nu)$$

almost always exists and result is a **Gaussian RV** with mean $\mathbb{E}\left[X(0)\right]\left(\int_{-\infty}^{\infty} s(t)\mathrm{d}t + \sum_{\nu=1}^{n} \alpha_\nu\right)$ and variance

$$\mathrm{Var}\left[\int_{-\infty}^{\infty} X(\omega, t)s(t)\mathrm{d}t + \sum_{\nu=1}^{n} \alpha_\nu X(\omega, t_\nu)\right] = \int_{-\infty}^{\infty} K_{XX}(\sigma)R_{ss}(\sigma)\mathrm{d}\sigma$$
$$+ \sum_{\nu=1}^{n} \sum_{\nu'=1}^{n} \alpha_\nu \alpha_{\nu'} K_{XX}(t_\nu - t_{\nu'}) + 2\sum_{\nu=1}^{n} \alpha_\nu \int_{-\infty}^{\infty} s(t)K_{XX}(t - t_\nu)\mathrm{d}t$$

Joint Distribution of Linear Functionals

- Even more: If $(X(t))$ is a Gaussian SP, then the joint distribution of a collection of linear functionals is jointly Gaussian!

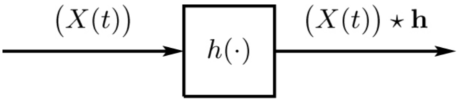
Let $s_j : \mathbb{R} \to \mathbb{R}$ measurable functions, $\alpha_{j,\nu} \in \mathbb{R}$. Then the linear functionals (RVs)

$$\omega \mapsto \int_{-\infty}^{\infty} X(\omega, t) s_j(t) \mathrm{d}t + \sum_{\nu=1}^{n_j} \alpha_{j,\nu} X(\omega, t_{j,\nu}), \quad j = 1, \ldots, m$$

of a measurable, Gaussian WSS SP $(X(t))$ are **jointly Gaussian.**

- Generally, second-order properties of linear functionals of measurable WSS SP can be characterized by the acf K_{XX}.
  - In the Gaussian case, the mean and covariance matrix completely specifies the joint distribution.

Filtering of WSS Processes $\int_{-\infty}^{\infty} X(\sigma)h(t-\sigma)\mathrm{d}\sigma$



(i) Passing a WSS SP through a stable filter produces a WSS SP with acf $K_{YY} = K_{XX} \star R_{hh}$, and $\mathbb{E}[X(t)Y(t+\tau)] = (K_{XX} \star h)(\tau)$
- $(X(t))$ and $(Y(t))$ are said *jointly wide-sense stationary*.

(ii) An input with PSD S_{XX}, then output PSD $S_{YY}(f) = S_{XX}(f)|\hat{h}(f)|^2$

(iii) If the input is Gaussian, then so is the output.
- Additionally, $(X(t))$ and $(Y(t))$ are jointly Gaussian.

- If h satisfies $\int_{-\infty}^{\infty} h^2(t)(1+t^2)\mathrm{d}t < \infty$, then the convolution is defined for all t and almost all ω (universality).

Wiener-Khinchin Theorem

Wiener-Khinchin Theorem

Let $(X(t))$ a measurable, centered, WSS SP with acf K_{XX} is passed through a stable filter with impulse response h, then the average power is

$$\text{Power of } X \star h = (K_{XX} \star R_{hh})(0) = \int_{-\infty}^{\infty} K_{XX}(\tau) R_{hh}(\tau) \mathrm{d}\tau$$

If additionally $(X(t))$ is of PSD S_{XX}, then

$$\text{Power of } X \star h = \int_{-\infty}^{\infty} S_{XX}(f) |\hat{h}(f)|^2 \mathrm{d}f$$

- This result can be used to show that PSD and operational PSD of a WSS SP are almost the same and both exist if one exits.

White Gaussian Noise

- Most important continuous-time SP, slightly differently defined:

White Gaussian Noise

$(N(t))$ is **white Gaussian noise of double-sided spectral density** $N_0/2$ **with respect to bandwidth** W if $(N(t))$ is measurable, stationary, centered, Gaussian SP with PSD satisfying

$$S_{NN}(f) = N_0/2 \qquad f \in [-W, W]$$

- Using the previous, many key properties can be derived, e.g.
  - $s(t)$ integrable W-bandlimited fct $\int_{-\infty}^{\infty} N(t)s(t)\mathrm{d}t \sim \mathcal{N}(0, N_0/2\|s\|^2)$
  - m such functions lead to jointly Gaussian RVs
  - If they are orthonormal, then the RVs are iid $\mathcal{N}(0, N_0/2)$
  - $K_{NN} \star s = N_0/2s$ and $\mathrm{Cov}\left[\int_{-\infty}^{\infty} N(\sigma)s(\sigma)\mathrm{d}\sigma, N(t)\right] = N_0/2s(t)$
- **Extension:** White noise in passband ($f \in [f_0 - W, f_0 + W]$)!

Outlook - Assignment

- Complex Gaussian and Circular Symmetry
- Continuous Time Stochastic Processes

Next lecture

Detection in White Noise; Non-coherent Detection and Nuisance Parameters

- Reading assignment: Chap 26-27
- Homework:
  - Problems in textbook: Exercises 24.1, 24.3, 24.4, 24.8, 25.1, 25.5, 25.10, and 25.11
  - Deadline: Dec 11