Complex RV and Processes, Energy, Power and PSD of QAM, Univariate Gaussian Distribution

Course: Foundations in Digital Communications

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3rd lecture

What did we do last lecture?

Outline - Motivation

- Base-band representations are complex-valued, thus we need
  - Complex RV and Processes (chap 17)
- How do the following concepts extend for complex signals?
  - Energy, Power, and PSD of QAM (chap 18)
- Let's have a look at an important distribution...
  - Univariate Gaussian Distribution (chap 19)

Motivation and Notation

- Complex RV (**CRV**) C defined on $(\Omega, \mathcal{F}, P)$ with $C : \Omega \to \mathbb{C}$
- CRV Z can be always seen as a **pair of two real RVs** X and Y by $Z = X + iY$
  - mean, variance follow accordingly
  - Not always recommended since complex RV in Digital Communicaitons often have an additional property which simplifies analysis (“proper” aka “circular symmetric”)
- Notation:
  - $(\cdot)^*$ denotes component complex conjugate
  - $(\cdot)^\dagger$ denotes Hermitian conjugate
  - If $A = A^\dagger$ matrix A is Hermitian, aka conjugate-symmetric, self-adjoint
- Convention: ’Vectors’ are usually column vectors

Definition of some Standard Terms

- Note that complex numbers cannot be sorted.
- RV W and Z are of **equal law** (i.e., $W \stackrel{\mathscr{L}}{=} Z$) iff

$$\mathbb{P}\left[\operatorname{Re}(W) \le x, \operatorname{Im}(W) \le y\right] = \mathbb{P}\left[\operatorname{Re}(Z) \le x, \operatorname{Im}(Z) \le y\right], \quad \forall x, y \in \mathbb{R}$$

- **Density function**, $z \in \mathbb{C}$ with $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$:

$$f_Z(z) \triangleq f_{\operatorname{Re}(Z),\operatorname{Im}(Z)}(\operatorname{Re}(z), \operatorname{Im}(z)) = \frac{\partial^2}{\partial x \partial y} \mathbb{P}\left[\operatorname{Re}(Z) \le x, \operatorname{Im}(Z) \le y\right]$$

- **Expectation:** $\mathbb{E}[Z] = \mathbb{E}[\operatorname{Re}(Z)] + i\mathbb{E}[\operatorname{Im}(Z)]$
- **Variance**

$$\operatorname{Var}[Z] \triangleq \mathbb{E}\left[|Z - \mathbb{E}[Z]|^2\right] = \ldots = \operatorname{Var}[\operatorname{Re}(Z)] + \operatorname{Var}[\operatorname{Im}(Z)]$$

Proper Complex Random Variables

- **Note:** $\mathrm{Var}[Z]$ is specified by the covariance matrix of $[X, Y]$ with $X = \mathrm{Re}(Z)$ and $Y = \mathrm{Im}(Z)$

$$\begin{bmatrix} \mathrm{Var}[\mathrm{Re}(Z)] & \mathrm{Cov}[\mathrm{Re}(Z)\,\mathrm{Im}(Z)] \\ \mathrm{Cov}[\mathrm{Re}(Z)\,\mathrm{Im}(Z)] & \mathrm{Var}[\mathrm{Im}(Z)] \end{bmatrix} = \begin{bmatrix} \mathrm{Var}[X] & \mathrm{Cov}[XY] \\ \mathrm{Cov}[XY] & \mathrm{Var}[Y] \end{bmatrix}$$

Definition: Proper CRV

A CRV is said to be **proper** if

(i) it is of zero mean,

(ii) it is of finite variance, and

(iii) $\mathbb{E}\left[Z^2\right] = 0$

- Note that $Z^2 = (X + iY)^2 = X^2 - Y^2 + i2XY$, thus

$$\mathbb{E}\left[Z^2\right] = 0 \quad \Leftrightarrow \quad \mathbb{E}\left[X^2\right] = \mathbb{E}\left[Y^2\right] \text{ and } \mathbb{E}[XY] = 0$$

Q: Does $\mathrm{Var}[Z]$ specify the covariance matrix of a (proper) CRV Z?

Covariance and Characteristic Fct of a CRV

- The covariance between CRVs is a complex scalar and not a real matrix.

Definition: Covariance

$$\mathrm{Cov}[Z, W] \triangleq \mathbb{E}\left[(Z - \mathbb{E}[Z])(W - \mathbb{E}[W])^*\right]$$

- For the characteristic function one can view a CRV as a pair of real RVs $\Phi_{X,Y} : \mathbb{R}^2 \to \mathbb{C}$, $\Phi_{X,Y}(\omega_1, \omega_2) = \mathbb{E}\left[e^{i(\omega_1 X + \omega_2 Y)}\right]$, $\omega_i \in \mathbb{R}$.

Definition: Characteristic Function

The characteristic function $\Phi_Z : \mathbb{C} \to \mathbb{C}$ of a CRV Z is defined as

$$\Phi_Z(\omega) \triangleq \mathbb{E}\left[e^{i\mathrm{Re}(\omega^* Z)}\right] = \mathbb{E}\left[e^{i(\mathrm{Re}(\omega)\mathrm{Re}(Z) + \mathrm{Im}(\omega)\mathrm{Im}(Z))}\right], \quad \omega \in \mathbb{C}$$

Transformation of Real Random Vectors (RV)

- Let $g : \mathcal{D} \to \mathcal{R}$, be a **one-to-one** mapping. $\mathcal{D}, \mathcal{R} \subseteq \mathbb{R}^n$.
  - g has continuous partial derivatives in $\mathcal{D}$
  - Jacobian determinant $\det\left(\frac{\partial g(\boldsymbol{x})}{\partial \boldsymbol{x}}\right) \neq 0$ for all $\boldsymbol{x} \in \mathcal{D}$

Theorem: Transformation of Real Random Vectors

Let $\boldsymbol{Y} = g(\boldsymbol{X})$ with RV $\boldsymbol{X}$ and $\mathbb{P}\left[X \in \mathcal{D}\right] = 1$, then

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = \frac{f_{\boldsymbol{X}}(g^{-1}(\boldsymbol{y}))}{\left|\det\left(\frac{\partial g(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)_{\boldsymbol{x}=g^{-1}(\boldsymbol{y})}\right|}$$

- The joint density $f_{R,\Theta}(r,\theta)$ of CRV Z with $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$ is given by

$$f_{R,\Theta}(r,\theta) = r f_Z(r\mathrm{e}^{i\theta}), \quad r > 0, \theta \in [-\pi, \pi).$$

Some Complex Analysis

- $g : \mathcal{D} \to \mathbb{C}$ is **differentiable** at $z_0 \in \mathcal{D}$ if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $h \in \mathbb{C}$ with $0 \leq |h| \leq \delta$ we have

$$\left| \frac{g(z_0 + h) - g(z_0)}{h} - g'(z_0) \right| < \varepsilon.$$

- g is **analytic** (or holomorphic) if g is differentiable at every $z \in \mathcal{D}$
- Analytic functions satisfy the **Cauchy-Riemann** equations:

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}, \qquad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}$$

with $u(x,y) = \mathrm{Re}\,(g(x+iy))$ and $v(x,y) = \mathrm{Im}\,(g(x+iy))$ and

$$g'(z) = \frac{\partial u(x,y)}{\partial x} + i\frac{\partial v(x,y)}{\partial x}\bigg|_{z=x+iy}$$

Transforming CRV

Theorem: Transforming CRV

$g : \mathcal{D} \to \mathcal{R}$ one-to-one mapping, analytic in $\mathcal{D}$, and derivative $\neq 0$ for all $z \in \mathcal{D}$. Let $W = g(Z)$ of CRV Z with $\mathbb{P}[Z \in \mathcal{D}] = 1$, then we have the density

$$f_W(w) = \frac{f_Z(g^{-1}(w))}{|g'(g^{-1}(w))|^2}, \qquad w \in \mathcal{R}$$

Proof:

- Consider CRV as pair of real RVs and apply previous theorem
- $g(x+iy) = u(x,y) + iv(x,y)$, thus $g : (x,y) \mapsto (u,v)$

$$\left|\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\right| = \left|\det\begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix}\right| = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |g'(x+iy)|^2$$

□

Complex Random Vectors

- Covariance matrix: $K_{\boldsymbol{ZZ}} \triangleq \mathbb{E}\left[(\boldsymbol{Z} - \mathbb{E}[\mathrm{Z}])(\boldsymbol{Z} - \mathbb{E}[\mathrm{Z}])^{\dagger}\right]$
- CRV $\boldsymbol{Z}$ is **proper** if it is of zero mean, finite variance, and

$$\mathbb{E}\left[\boldsymbol{Z}\boldsymbol{Z}^{T}\right] = \mathbf{0}$$

- Linear transformation $\boldsymbol{Y} = A\boldsymbol{Z}$ of proper CRV $\boldsymbol{Z}$ are proper.

$$\mathbb{E}\left[\boldsymbol{Y}\boldsymbol{Y}^{T}\right] = \mathbb{E}\left[A\boldsymbol{Z}(A\boldsymbol{Z})^{T}\right] = \mathbb{E}\left[A\boldsymbol{Z}\boldsymbol{Z}^{T}A^{T}\right] = A\mathbb{E}\left[\boldsymbol{Z}\boldsymbol{Z}^{T}\right]A^{T} = \mathbf{0}$$

Complex Stochastic Processes (CSP)

- **Complex stochastic process:** collection of CRV $(Z(t), t \in \mathcal{T})$ defined on a common probability space $(\Omega, \mathcal{F}, P)$
  - Definition of strongly and weakly stationary directly extend while for the second moment the second term is conjugate complex.
- CSP $(\mathbf{Z}_\nu)$ is **proper** if it is centered, finite variance, and

$$\mathbb{E}\left[\mathbf{Z}_\nu \mathbf{Z}_{\nu'}\right] = 0 \qquad \nu, \nu' \in \mathbb{Z}$$

- **Autocovariance function** of a WSS CSP $(\mathbf{Z}_\nu)$, $\eta \in \mathbb{Z}$,

$$K_{ZZ}(\eta) \triangleq \mathrm{Cov}\left[Z_{\nu+\eta}, Z_\nu\right] = \mathbb{E}\left[(Z_{\nu+\eta} - \mathbb{E}\left[Z_1\right])(Z_\nu - \mathbb{E}\left[Z_1\right])^*\right]$$

  - **Power spectral density** defined by $K_{ZZ}(\eta) = \int_{-1/2}^{1/2} S_{ZZ}(\theta) e^{i2\pi\eta\theta} \, d\theta$

Let's take a break!

Energy, Power and PSD of QAM

- **QAM signal** with complex symbols C_ℓ, $W/2$ bandlimited pulse g, carrier frequency $f_c > W/2$, and real amplitude A

$$X(t) = 2\mathrm{Re}\left(X_{BB}(t)\mathrm{e}^{i2\pi f_c t}\right), \quad X_{BB}(t) = A\sum_{\ell} C_\ell g(t - \ell T_s)$$

- Most of the previous concepts directly transfer from real-valued to complex-valued, new aspects:
  - How is the relationship between passband and baseband?
  - Where to put the conjugate complex operation?

Energy of QAM

- Energy E of transmitted signal $X(t)$

$$E \triangleq \mathbb{E}\left[\int_{-\infty}^{\infty} X^2(t)\mathrm{d}t\right] = 2\mathbb{E}\left[\int_{-\infty}^{\infty} |X_{BB}(t)|^2\mathrm{d}t\right]$$

  since $||x_{PB}||^2 = 2||x_{BB}||^2$ and $g(\cdot)$ bandlimited to $W/2$

$$\mathbb{E}\left[\int_{-\infty}^{\infty} |X_{BB}(t)|^2\mathrm{d}t\right] = \int_{-\infty}^{\infty} \mathbb{E}\left[(A\sum_{\ell=1}^{N} C_\ell g(t-\ell T_s))(A\sum_{\ell'=1}^{N} C_{\ell'} g(t-\ell' T_s))^*\right]\mathrm{d}t$$

$$= A^2 \sum_{\ell=1}^{N}\sum_{\ell'=1}^{N} \mathbb{E}\left[C_\ell C_{\ell'}^*\right] \underbrace{\int_{-\infty}^{\infty} g(t-\ell T_s) g^*(t-\ell' T_s)\mathrm{d}t}_{=R_{gg}((\ell'-\ell)T_s)}$$

- $E = 2A^2||g||^2 \sum_{\ell=1}^{N} \mathbb{E}\left[|C_\ell|^2\right]$
  - if $\{C_\ell\}$ are zero mean and uncorrelated, or
  - if pulses are orthogonal by time-shifts of integer multiples of T_s

Power of QAM

Assumptions:

- infinite sequence of complex symbols ($N \to \infty$)
- pulse g satisfies decay condition $|g(t)| \leq \frac{\beta}{1+|t/T_s|^{1+\alpha}}$, $\alpha, \beta > 0$
- sequence $\{C_\ell\}$ is bounded

Power in QAM

$$\lim_{T\to\infty} \frac{1}{2T} \mathbb{E}\left[\int_{-T}^{T} X^2(t)\mathrm{d}t\right] = \mathbf{2} \lim_{T\to\infty} \frac{1}{2T} \mathbb{E}\left[\int_{-T}^{T} |X_{BB}^2(t)|^2 \mathrm{d}t\right]$$

- Relation does not hold for $T < \infty$ since $X(t)$ is not bandlimited
- If CSP (C_ℓ) is additionally zero-mean and WSS, then

$$\lim_{T\to\infty} \frac{1}{2T} \mathbb{E}\left[\int_{-T}^{T} X^2(t)\mathrm{d}t\right] = \frac{2A^2}{T_S} \sum_{m=-\infty}^{\infty} K_{CC}(m) R_{gg}^{*}(mT_s)$$

Operational PSD of CSP

Definition

The CSP $Z(t)$ is of **operational power spectral density** $S_{ZZ}(f)$ if

(i) $Z(t)$ is measurable (real and complex part are measurable SP);

(ii) the function $S_{ZZ} : \mathbb{R} \to \mathbb{R}$ is integrable; and

(iii) for every absolute integrable complex-valued function $h : \mathbb{R} \to \mathbb{C}$ the average power at the output of the filter with input $Z(t)$ is given by

$$\text{Power of } Z \star h = \int_{-\infty}^{\infty} |\hat{h}(f)|^2 S_{ZZ}(f)\,\mathrm{d}f$$

- Difference to real-valued SP
  - $\rightarrow$ (ii): operational PSD needs not be symmetric
  - $\rightarrow$ (iii) has to hold for all complex-valued filters

QAM Relationship between Passband and Baseband

Relationship between operational PSD S_{XX} of a real QAM signal and the operational PSD S_{BB} of the corresponding baseband CSP $X_{BB}(t)$

$$S_{XX}(f) = S_{BB}(|f| - f_c), \quad f \in \mathbb{R}.$$

- PAM: g is $W/2$ bandlimited $\Rightarrow S_{BB}(f) = 0$ for $|f| > W/2$.
  - For every h: $g \star h = (g \star \text{LPF}_{W/2}) \star h = g \star (\text{LPF}_{W/2} \star h) = g \star h'$
  - Baseb. representation of passb. filter $\hat{h}'_{BB}(f) = \hat{h}(f + f_c) I\{|f| \leq W/2\}$

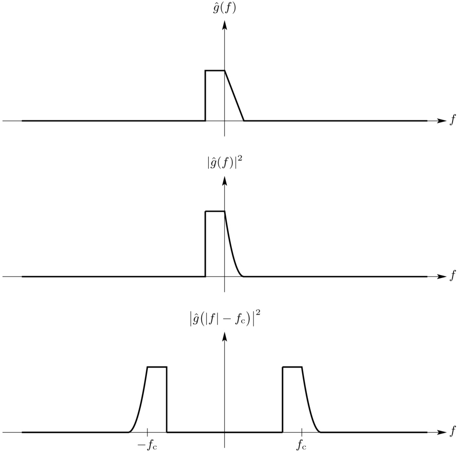
$$\text{Power in } X \star h = 2 \,\text{Power in } X_{BB} \star h'_{BB} = 2 \int_{-\infty}^{\infty} S_{BB}(f) |\hat{h}'_{BB}(f)|^2 \mathrm{d}f$$

$$= 2 \int_{-\infty}^{\infty} S_{BB}(f) |\hat{h}'(f + f_c)|^2 \mathrm{d}f = 2 \int_{-\infty}^{\infty} S_{BB}(\tilde{f} - f_c) |\hat{h}(\tilde{f})|^2 \mathrm{d}\tilde{f}$$

$$= \int_{-\infty}^{\infty} S_{BB}(\tilde{f} - f_c)\left(|\hat{h}(\tilde{f})|^2 + |\hat{h}(-\tilde{f})|^2\right) \mathrm{d}\tilde{f} = \int_{-\infty}^{\infty} S_{BB}(|f| - f_c) |\hat{h}(f)|^2 \mathrm{d}f$$

QAM with (C_ℓ) uncorrelated and zero mean

$$S_{XX}(f) = \frac{A^2}{T_s} \sum_{m=-\infty}^{\infty} K_{CC}(m) \mathrm{e}^{i2\pi(|f|-f_c)mT_s} |\hat{g}(|f|-f_c)|^2$$



Univariate Gaussian Distribution

Motivation

- Noise is often modeled as a Gaussian stochastic process
  - Strongly motivated by *Central limit theorem*; when many small independent disturbances add up, then distribution converges to Gaussian distribution (good approximation for finite terms).
  - Mathematical convenience - often amenable to analysis.

Definition: Standard Gaussian Distribution

RV W is a **standard Gaussian** if its density is given by

$$f_W(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}, \quad w \in \mathbb{R}$$

- Standard Gaussian random variable is symmetric $\Rightarrow \mathbb{E}[W] = 0$
- From some simple analysis it follows that the variance is one.

Gaussian Random Variables

- Let W be standard Gaussian, then RV X is said
  - **centered Gaussian** if $X = aW$ for some $a \in \mathbb{R}$
  - **Gaussian** if $X = aW + b$ for some $a, b \in \mathbb{R}$

5-minute Exercise: Closed w.r.t. affine transformations

Show that if X is Gaussian, then $\alpha X + \beta$ is Gaussian.

- Mean of X: $\mathbb{E}[X] = a\mathbb{E}[W] + b = b$
- Variance of X: $\mathrm{Var}[X] = \mathbb{E}\left[X^2\right] - b^2 = a^2\mathbb{E}\left[W^2\right] + 2a\mathbb{E}[W] = a^2$
- There exists only one RV X with Gaussian distribution of a given mean μ and variance σ^2. Notation: $X \sim \mathcal{N}(\nu, \sigma^2)$
  - Customary to require σ non-negative, then σ **standard deviation**

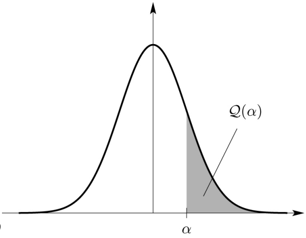
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x \in \mathbb{R}$$

Cumulative Distribution Function

- Cumulative Distribution Function of standard Gaussian RV W

$$F_W(w) = \mathbb{P}[W \leq w] = \int_{-\infty}^{w} f_W(\xi)\mathrm{d}\xi = \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\xi^2}{2}} \mathrm{d}\xi$$

- Q-function $Q(\alpha) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} \mathrm{e}^{-\xi/2}\mathrm{d}\xi$
  - $\mathbb{P}[a \leq W \leq b] = Q(a) - Q(b)$
- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then
  - $\mathbb{P}[X \leq a] = Q\left(\frac{a-\mu}{\sigma}\right)$



- Craig's formula: $Q(\alpha) = \frac{1}{\pi} \int_0^{\pi/2} \mathrm{e}^{-\frac{\alpha^2}{2\sin^2\phi}} \mathrm{d}\phi$
- Upper and lower bounds on the Q-function

$$\frac{1}{\sqrt{2\pi\alpha^2}} \mathrm{e}^{-\alpha^2/2}(1 - \alpha^{-2}) \leq Q(\alpha) \leq \min\left\{\frac{1}{\sqrt{2\pi\alpha^2}} \mathrm{e}^{-\alpha^2/2}, \frac{1}{2}\mathrm{e}^{-\alpha^2/2}\right\}$$

Characteristic Function

- Characteristic fct of a RV X: $\Phi_X(\omega) = \mathbb{E}\left[e^{i\omega X}\right] = \int_{-\infty}^{\infty} f_X(x) e^{i\omega x} dx$

$X \& Y$ are independent RVs $\Rightarrow \Phi_{X+Y}(\omega) = \Phi_X(\omega)\Phi_Y(\omega)$

- For $X \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow \Phi(\omega) = e^{i\omega\mu - \frac{1}{2}\omega^2\sigma^2}$

Sums of independent Gaussians

Let $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and $Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$ be independent Gaussian RV, then $X + Y \sim \mathcal{N}(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$.

Proof:

$$\Phi_{X+Y}(\omega) = e^{i\omega\mu_x - \frac{1}{2}\omega^2\sigma_x^2} e^{i\omega\mu_y - \frac{1}{2}\omega^2\sigma_y^2} = e^{i\omega(\mu_x+\mu_y) - \frac{1}{2}\omega^2(\sigma_x^2+\sigma_y^2)}$$

Related Distributions

- **Central χ^2-distribution** with n degrees of freedom χ_n^2

$$X_1, \ldots, X_n \sim \text{IID}\mathcal{N}(0,1) \Rightarrow \sum_{j=1}^{n} X_j^2 \sim \chi_n^2$$

$$f_{\chi_n^2}(x) = \frac{1}{2^{n/2}\Gamma(n/2)} \mathrm{e}^{-x/2} x^{(n/2)-1}, \quad , x > 0$$

  with $\Gamma(\xi) = \int_0^\infty \mathrm{e}^{-t} t^{\xi-1} \mathrm{d}t, \xi > 0$.
- Others
  - Generalized Rayleigh distribution: $\sqrt{\chi_n^2}$
  - Rayleigh distribution: $n = 2$
  - Noncentral χ_n^2 distribution using $X_j \sim \text{IID}\mathcal{N}(\nu_j, \sigma^2)$
  - Generalized Rice distribution, Rice distribution ($\sqrt{\text{Noncentral } \chi_n^2}$)

App: Definitions of Convergence

Good to know: Let RVs $X_1, X_2, \ldots$ be defined on $(\Omega, \mathcal{F}, P)$ we say sequence converges to X...

- **with probability one** or **almost surely** if

$$\mathbb{P}\left[\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right] = 1;$$

- **in probability** if

$$\lim_{n\to\infty} \mathbb{P}\left[|X_n - X| \geq \varepsilon\right] = 0, \quad \varepsilon > 0;$$

- **in mean square** if

$$\lim_{n\to\infty} \mathbb{E}\left[(X_n - X)^2\right] = 0;$$

- It can be shown that convergence in mean-square and almost-sure convergence implies convergence in probability.
- Another type of convergence is **weak convergence** where the cumulative distributions $F_1, F_2, \ldots$ converge $F(\cdot)$ at every point $\xi \in \mathbb{R}$ at which $F(\cdot)$ is continuous.

Outlook - Assignment

- Complex RV and Processes
- Operational Power Spectrum Density of QAM
- Univariate Gaussian Distribution

Next lecture

Binary and Multiple Hypothesis Testing

- Reading Assignment: Chap 20-21
- Homework:
  - Problems in textbook: Exercise 17.1, 17.2, 17.6, 17.7, 17.12, 18.2, 18.6, 18.8, 19.7, and 19.11
  - Deadline: Nov 26