

Complex RV and Processes, Energy, Power and PSD of QAM, Univariate Gaussian Distribution

Course: Foundations in Digital Communications

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3rd lecture

**What did we do last
lecture?**

Outline - Motivation

- Base-band representations are complex-valued, thus we need
 - Complex RV and Processes (chap 17)
- How do the following concepts extend for complex signals?
 - Energy, Power, and PSD of QAM (chap 18)
- Let's have a look at an important distribution...
 - Univariate Gaussian Distribution (chap 19)

Motivation and Notation

- Complex RV (**CRV**) C defined on (Ω, \mathcal{F}, P) with $C : \Omega \rightarrow \mathbb{C}$
- CRV Z can be always seen as a **pair of two real RVs** X and Y by $Z = X + iY$
 - mean, variance follow accordingly
 - Not always recommended since complex RV in Digital Communicaitons often have an additional property which simplifies analysis (“proper” aka “circular symmetric”)
- Notation:
 - $(\cdot)^*$ denotes component complex conjugate
 - $(\cdot)^\dagger$ denotes Hermitian conjugate
 - If $A = A^\dagger$ matrix A is Hermitian, aka conjugate-symmetric, self-adjoint
- Convention: ‘Vectors’ are usually column vectors

Definition of some Standard Terms

- Note that complex numbers cannot be sorted.
- RV W and Z are of **equal law** (i.e., $W \stackrel{\mathcal{L}}{=} Z$) iff

$$\mathbb{P} [\operatorname{Re}(W) \leq x, \operatorname{Im}(W) \leq y] = \mathbb{P} [\operatorname{Re}(Z) \leq x, \operatorname{Im}(Z) \leq y], \quad \forall x, y \in \mathbb{R}$$

- **Density function**, $z \in \mathbb{C}$ with $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$:

$$f_Z(z) \triangleq f_{\operatorname{Re}(Z), \operatorname{Im}(Z)}(\operatorname{Re}(z), \operatorname{Im}(z)) = \frac{\partial^2}{\partial x \partial y} \mathbb{P} [\operatorname{Re}(Z) \leq x, \operatorname{Im}(Z) \leq y]$$

- **Expectation:** $\mathbb{E}[Z] = \mathbb{E}[\operatorname{Re}(Z)] + i\mathbb{E}[\operatorname{Im}(Z)]$
- **Variance**

$$\operatorname{Var}[Z] \triangleq \mathbb{E}[|Z - \mathbb{E}[Z]|^2] = \dots = \operatorname{Var}[\operatorname{Re}(Z)] + \operatorname{Var}[\operatorname{Im}(Z)]$$

Proper Complex Random Variables

- **Note:** $\text{Var}[Z]$ is specified by the covariance matrix of $[X, Y]$ with $X = \text{Re}(Z)$ and $Y = \text{Im}(Z)$

$$\begin{bmatrix} \text{Var}[\text{Re}(Z)] & \text{Cov}[\text{Re}(Z) \text{Im}(Z)] \\ \text{Cov}[\text{Re}(Z) \text{Im}(Z)] & \text{Var}[\text{Im}(Z)] \end{bmatrix} = \begin{bmatrix} \text{Var}[X] & \text{Cov}[XY] \\ \text{Cov}[XY] & \text{Var}[Y] \end{bmatrix}$$

Definition: Proper CRV

A CRV is said to be **proper** if

- it is of zero mean,
- it is of finite variance, and
- $\mathbb{E}[Z^2] = 0$

- Note that $Z^2 = (X + iY)^2 = X^2 - Y^2 + i2XY$, thus

$$\mathbb{E}[Z^2] = 0 \quad \Leftrightarrow \quad \mathbb{E}[X^2] = \mathbb{E}[Y^2] \quad \text{and} \quad \mathbb{E}[XY] = 0$$

Q: Does $\text{Var}[Z]$ specify the covariance matrix of a (proper) CRV Z ?

Covariance and Characteristic Fct of a CRV

- The covariance between CRVs is a complex scalar and not a real matrix.

Definition: Covariance

$$\text{Cov}[Z, W] \triangleq \mathbb{E}[(Z - \mathbb{E}[Z])(W - \mathbb{E}[W])^*]$$

- For the characteristic function one can view a CRV as a pair of real RVs $\Phi_{X,Y} : \mathbb{R}^2 \rightarrow \mathbb{C}$, $\Phi_{X,Y}(\omega_1, \omega_2) = \mathbb{E}[e^{i(\omega_1 X + \omega_2 Y)}]$, $\omega_i \in \mathbb{R}$.

Definition: Characteristic Function

The characteristic function $\Phi_Z : \mathbb{C} \rightarrow \mathbb{C}$ of a CRV Z is defined as

$$\Phi_Z(\omega) \triangleq \mathbb{E}[e^{i\text{Re}(\omega^* Z)}] = \mathbb{E}[e^{i(\text{Re}(\omega)\text{Re}(Z) + \text{Im}(\omega)\text{Im}(Z))}], \quad \omega \in \mathbb{C}$$

Transformation of Real Random Vectors (RV)

- Let $g : \mathcal{D} \rightarrow \mathcal{R}$, be a **one-to-one** mapping. $\mathcal{D}, \mathcal{R} \subseteq \mathbb{R}^n$.
 - g has continuous partial derivatives in \mathcal{D}
 - Jacobian determinant $\det\left(\frac{\partial g(x)}{\partial x}\right) \neq 0$ for all $x \in \mathcal{D}$

Theorem: Transformation of Real Random Vectors

Let $Y = g(X)$ with RV X and $\mathbb{P}[X \in \mathcal{D}] = 1$, then

$$f_Y(\mathbf{y}) = \frac{f_X(g^{-1}(\mathbf{y}))}{\left| \det\left(\frac{\partial g(x)}{\partial x}\right)_{x=g^{-1}(\mathbf{y})} \right|}$$

- The joint density $f_{R,\Theta}(r, \theta)$ of CRV Z with $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ is given by

$$f_{R,\Theta}(r, \theta) = r f_Z(re^{i\theta}), \quad r > 0, \theta \in [-\pi, \pi).$$

Some Complex Analysis

- $g : \mathcal{D} \rightarrow \mathbb{C}$ is **differentiable** at $z_0 \in \mathcal{D}$ if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $h \in \mathbb{C}$ with $0 \leq |h| \leq \delta$ we have

$$\left| \frac{g(z_0 + h) - g(z_0)}{h} - g'(z_0) \right| < \varepsilon.$$

- g is **analytic** (or holomorphic) if g is differentiable at every $z \in \mathcal{D}$
- Analytic functions satisfy the **Cauchy-Riemann** equations:

$$\frac{\partial u(x, y)}{\partial x} = \frac{\partial v(x, y)}{\partial y}, \quad \frac{\partial u(x, y)}{\partial y} = -\frac{\partial v(x, y)}{\partial x}$$

with $u(x, y) = \operatorname{Re}(g(x + iy))$ and $v(x, y) = \operatorname{Im}(g(x + iy))$ and

$$g'(z) = \frac{\partial u(x, y)}{\partial x} + i \frac{\partial v(x, y)}{\partial x} \Big|_{z=x+iy}$$

Transforming CRV

Theorem: Transforming CRV

$g : \mathcal{D} \rightarrow \mathcal{R}$ one-to-one mapping, analytic in \mathcal{D} , and derivative $\neq 0$ for all $z \in \mathcal{D}$. Let $W = g(Z)$ of CRV Z with $\mathbb{P}[Z \in \mathcal{D}] = 1$, then we have the density

$$f_W(w) = \frac{f_Z(g^{-1}(w))}{|g'(g^{-1}(w))|^2}, \quad w \in \mathcal{R}$$

Proof:

- Consider CRV as pair of real RVs and apply previous theorem
- $g(x + iy) = u(x, y) + iv(x, y)$, thus $g : (x, y) \mapsto (u, v)$

$$\left| \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \right| = \left| \det \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix} \right| = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 = |g'(x + iy)|^2$$



Complex Random Vectors

- Covariance matrix: $K_{ZZ} \triangleq \mathbb{E}[(\mathbf{Z} - \mathbb{E}[\mathbf{Z}])(\mathbf{Z} - \mathbb{E}[\mathbf{Z}])^\dagger]$
- CRV \mathbf{Z} is **proper** if it is of zero mean, finite variance, and

$$\mathbb{E}[\mathbf{Z}\mathbf{Z}^T] = \mathbf{0}$$

- Linear transformation $\mathbf{Y} = \mathbf{A}\mathbf{Z}$ of proper CRV \mathbf{Z} are proper.

$$\mathbb{E}[\mathbf{Y}\mathbf{Y}^T] = \mathbb{E}[\mathbf{A}\mathbf{Z}(\mathbf{A}\mathbf{Z})^T] = \mathbb{E}[\mathbf{A}\mathbf{Z}\mathbf{Z}^T\mathbf{A}^T] = \mathbf{A}\mathbb{E}[\mathbf{Z}\mathbf{Z}^T]\mathbf{A}^T = \mathbf{0}$$

Complex Stochastic Processes (CSP)

- **Complex stochastic process:** collection of CRV $(Z(t), t \in \mathcal{T})$ defined on a common probability space (Ω, \mathcal{F}, P)
 - Definition of strongly and weakly stationary directly extend while for the second moment the second term is conjugate complex.
- CSP (Z_ν) is **proper** if it is centered, finite variance, and

$$\mathbb{E}[Z_\nu Z_{\nu'}] = 0 \quad \nu, \nu' \in \mathbb{Z}$$

- **Autocovariance function** of a WSS CSP (Z_ν) , $\eta \in \mathbb{Z}$,

$$K_{ZZ}(\eta) \triangleq \text{Cov}[Z_{\nu+\eta}, Z_\nu] = \mathbb{E}[(Z_{\nu+\eta} - \mathbb{E}[Z_1])(Z_\nu - \mathbb{E}[Z_1])^*]$$

- **Power spectral density** defined by $K_{ZZ}(\eta) = \int_{-1/2}^{1/2} S_{ZZ}(\theta) e^{i2\pi\eta\theta} d\theta$

Let's take a break!

Energy, Power and PSD of QAM

- **QAM signal** with complex symbols C_ℓ , $W/2$ bandlimited pulse g , carrier frequency $f_c > W/2$, and real amplitude A

$$X(t) = 2\text{Re}\left(X_{BB}(t)e^{i2\pi f_c t}\right), \quad X_{BB}(t) = A \sum_{\ell} C_\ell g(t - \ell T_s)$$

- Most of the previous concepts directly transfer from real-valued to complex-valued, new aspects:
 - How is the relationship between passband and baseband?
 - Where to put the conjugate complex operation?

Energy of QAM

- Energy E of transmitted signal $X(t)$

$$E \triangleq \mathbb{E} \left[\int_{-\infty}^{\infty} X^2(t) dt \right] = 2 \mathbb{E} \left[\int_{-\infty}^{\infty} |X_{BB}(t)|^2 dt \right]$$

since $\|x_{PB}\|^2 = 2\|x_{BB}\|^2$ and $g(\cdot)$ bandlimited to $W/2$

$$\begin{aligned} \mathbb{E} \left[\int_{-\infty}^{\infty} |X_{BB}(t)|^2 dt \right] &= \int_{-\infty}^{\infty} \mathbb{E} \left[\left(A \sum_{\ell=1}^N C_{\ell} g(t - \ell T_s) \right) \left(A \sum_{\ell'=1}^N C_{\ell'} g(t - \ell' T_s) \right)^* \right] dt \\ &= A^2 \sum_{\ell=1}^N \sum_{\ell'=1}^N \mathbb{E} [C_{\ell} C_{\ell'}^*] \underbrace{\int_{-\infty}^{\infty} g(t - \ell T_s) g^*(t - \ell' T_s) dt}_{=R_{gg}((\ell' - \ell)T_s)} \end{aligned}$$

- $E = 2A^2 \|g\|^2 \sum_{\ell=1}^N \mathbb{E} [|C_{\ell}|^2]$
 - if $\{C_{\ell}\}$ are zero mean and uncorrelated, or
 - if pulses are orthogonal by time-shifts of integer multiples of T_s

Power of QAM

Assumptions:

- infinite sequence of complex symbols ($N \rightarrow \infty$)
- pulse g satisfies decay condition $|g(t)| \leq \frac{\beta}{1+|t/T_s|^{1+\alpha}}$, $\alpha, \beta > 0$
- sequence $\{C_\ell\}$ is bounded

Power in QAM

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \mathbb{E} \left[\int_{-T}^T X^2(t) dt \right] = 2 \lim_{T \rightarrow \infty} \frac{1}{2T} \mathbb{E} \left[\int_{-T}^T |X_{BB}^2(t)|^2 dt \right]$$

- Relation does not hold for $T < \infty$ since $X(t)$ is not bandlimited
- If CSP (C_ℓ) is additionally zero-mean and WSS, then

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \mathbb{E} \left[\int_{-T}^T X^2(t) dt \right] = \frac{2A^2}{T_s} \sum_{m=-\infty}^{\infty} K_{CC}(m) R_{gg}^*(mT_s)$$

Operational PSD of CSP

Definition

The CSP $Z(t)$ is of **operational power spectral density** $S_{ZZ}(f)$ if

- (i) $Z(t)$ is measurable (real and complex part are measurable SP);
- (ii) the function $S_{ZZ} : \mathbb{R} \rightarrow \mathbb{R}$ is integrable; and
- (iii) for every absolute integrable complex-valued function $h : \mathbb{R} \rightarrow \mathbb{C}$ the average power at the output of the filter with input $Z(t)$ is given by

$$\text{Power of } Z \star h = \int_{-\infty}^{\infty} |\hat{h}(f)|^2 S_{ZZ}(f) df$$

- Difference to real-valued SP
 - (ii): operational PSD needs not be symmetric
 - (iii) has to hold for all complex-valued filters

QAM Relationship between Passband and Baseband

Relationship between operational PSD S_{XX} of a real QAM signal and the operational PSD S_{BB} of the corresponding baseband CSP $X_{BB}(t)$

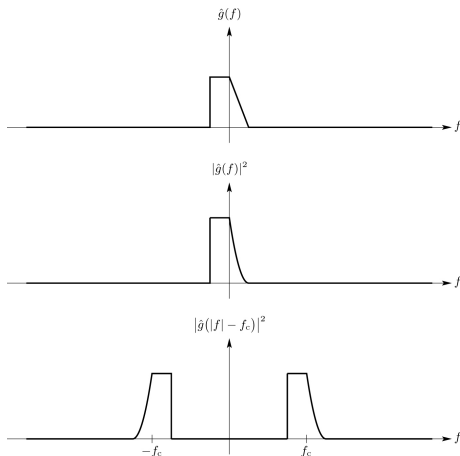
$$S_{XX}(f) = S_{BB}(|f| - f_c), \quad f \in \mathbb{R}.$$

- PAM: g is $W/2$ bandlimited $\Rightarrow S_{BB}(f) = 0$ for $|f| > W/2$.
 - For every h : $g \star h = (g \star \text{LPF}_{W/2}) \star h = g \star (\text{LPF}_{W/2} \star h) = g \star h'$
 - Baseb. representation of passb. filter $\hat{h}'_{BB}(f) = \hat{h}(f + f_c)I\{|f| \leq W/2\}$

$$\begin{aligned} \text{Power in } X \star h &= 2 \text{ Power in } X_{BB} \star h'_{BB} = 2 \int_{-\infty}^{\infty} S_{BB}(f) |\hat{h}'_{BB}(f)|^2 df \\ &= 2 \int_{-\infty}^{\infty} S_{BB}(f) |\hat{h}'(f + f_c)|^2 df = 2 \int_{-\infty}^{\infty} S_{BB}(\tilde{f} - f_c) |\hat{h}(\tilde{f})|^2 d\tilde{f} \\ &= \int_{-\infty}^{\infty} S_{BB}(\tilde{f} - f_c) (|\hat{h}(\tilde{f})|^2 + |\hat{h}(-\tilde{f})|^2) d\tilde{f} = \int_{-\infty}^{\infty} S_{BB}(|f| - f_c) |\hat{h}(f)|^2 df \end{aligned}$$

QAM with (C_ℓ) uncorrelated and zero mean

$$S_{XX}(f) = \frac{A^2}{T_s} \sum_{m=-\infty}^{\infty} K_{CC}(m) e^{i2\pi(|f|-f_c)mT_s} |\hat{g}(|f|-f_c)|^2$$



Univariate Gaussian Distribution

Motivation

- Noise is often modeled as a Gaussian stochastic process
 - Strongly motivated by *Central limit theorem*; when many small independent disturbances add up, then distribution converges to Gaussian distribution (good approximation for finite terms).
 - Mathematical convenience - often amenable to analysis.

Definition: Standard Gaussian Distribution

RV W is a **standard Gaussian** if its density is given by

$$f_W(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}, \quad w \in \mathbb{R}$$

- Standard Gaussian random variable is symmetric $\Rightarrow \mathbb{E}[W] = 0$
- From some simple analysis it follows that the variance is one.

Gaussian Random Variables

- Let W be standard Gaussian, then RV X is said
 - **centered Gaussian** if $X = aW$ for some $a \in \mathbb{R}$
 - **Gaussian** if $X = aW + b$ for some $a, b \in \mathbb{R}$

5-minute Exercise: Closed w.r.t. affine transformations

Show that if X is Gaussian, then $\alpha X + \beta$ is Gaussian.

- Mean of X : $\mathbb{E}[X] = a\mathbb{E}[W] + b = b$
- Variance of X : $\text{Var}[X] = \mathbb{E}[X^2] - b^2 = a^2\mathbb{E}[W^2] + 2a\mathbb{E}[W] = a^2$
- There exists only one RV X with Gaussian distribution of a given mean μ and variance σ^2 . Notation: $X \sim \mathcal{N}(\mu, \sigma^2)$
 - Customary to require σ non-negative, then σ **standard deviation**

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

Cumulative Distribution Function

- Cumulative Distribution Function of standard Gaussian RV W

$$F_W(w) = \mathbb{P}[W \leq w] = \int_{-\infty}^w f_W(\xi) d\xi = \int_{-\infty}^w \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi$$

- Q-function $Q(\alpha) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\xi^2/2} d\xi$

- $\mathbb{P}[a \leq W \leq b] = Q(a) - Q(b)$

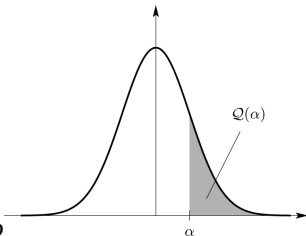
- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then

- $\mathbb{P}[X \leq a] = Q\left(\frac{a-\mu}{\sigma}\right)$

- Craig's formula: $Q(\alpha) = \frac{1}{\pi} \int_0^{\pi/2} e^{-\frac{\alpha^2}{2 \sin^2 \phi}} d\phi$

- Upper and lower bounds on the Q-function

$$\frac{1}{\sqrt{2\pi\alpha^2}} e^{-\alpha^2/2} (1 - \alpha^{-2}) \leq Q(\alpha) \leq \min \left\{ \frac{1}{\sqrt{2\pi\alpha^2}} e^{-\alpha^2/2}, \frac{1}{2} e^{-\alpha^2/2} \right\}$$



Characteristic Function

- Characteristic fct of a RV X : $\Phi_X(\omega) = \mathbb{E} \left[e^{i\omega X} \right] = \int_{-\infty}^{\infty} f_X(x) e^{i\omega x} dx$

X & Y are independent RVs $\Rightarrow \Phi_{X+Y}(\omega) = \Phi_X(\omega)\Phi_Y(\omega)$

- For $X \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow \Phi(\omega) = e^{i\omega\mu - \frac{1}{2}\omega^2\sigma^2}$

Sums of independent Gaussians

Let $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and $Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$ be independent Gaussian RV, then $X + Y \sim \mathcal{N}(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$.

Proof:

$$\Phi_{X+Y}(\omega) = e^{i\omega\mu_x - \frac{1}{2}\omega^2\sigma_x^2} e^{i\omega\mu_y - \frac{1}{2}\omega^2\sigma_y^2} = e^{i\omega(\mu_x + \mu_y) - \frac{1}{2}\omega^2(\sigma_x^2 + \sigma_y^2)}$$

Related Distributions

- **Central χ^2 -distribution** with n degrees of freedom χ_n^2

$$X_1, \dots, X_n \sim \text{IID}\mathcal{N}(0, 1) \Rightarrow \sum_{j=1}^n X_j^2 \sim \chi_n^2$$

$$f_{\chi_n^2}(x) = \frac{1}{2^{n/2}\Gamma(n/2)} e^{-x/2} x^{(n/2)-1}, \quad x > 0$$

with $\Gamma(\xi) = \int_0^\infty e^{-t} t^{\xi-1} dt, \xi > 0$.

- Others

- Generalized Rayleigh distribution: $\sqrt{\chi_n^2}$
- Rayleigh distribution: $n = 2$
- Noncentral χ_n^2 distribution using $X_j \sim \text{IID}\mathcal{N}(v_j, \sigma^2)$
- Generalized Rice distribution, Rice distribution ($\sqrt{\text{Noncentral } \chi_n^2}$)

App: Definitions of Convergence

Good to know: Let RVs X_1, X_2, \dots be defined on (Ω, \mathcal{F}, P) we say sequence converges to X ...

- **with probability one or almost surely** if

$$\mathbb{P} \left[\left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \right\} \right] = 1;$$

- **in probability** if

$$\lim_{n \rightarrow \infty} \mathbb{P} [|X_n - X| \geq \varepsilon] = 0, \quad \varepsilon > 0;$$

- **in mean square** if

$$\lim_{n \rightarrow \infty} \mathbb{E} [(X_n - X)^2] = 0;$$

- It can be shown that convergence in mean-square and almost-sure convergence implies convergence in probability.
- Another type of convergence is **weak convergence** where the cumulative distributions F_1, F_2, \dots converge $F(\cdot)$ at every point $\xi \in \mathbb{R}$ at which $F(\cdot)$ is continuous.

Outlook - Assignment

- Complex RV and Processes
- Operational Power Spectrum Density of QAM
- Univariate Gaussian Distribution

Next lecture

Binary and Multiple Hypothesis Testing

- Reading Assignment: Chap 20-21
- Homework:
 - Problems in textbook: Exercise 17.1, 17.2, 17.6, 17.7, 17.12, 18.2, 18.6, 18.8, 19.7, and 19.11
 - Deadline: Nov 27