Energy and Power in PAM, Operational PSD, QAM

Course: Foundations in Digital Communications

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2nd lecture

What did we do last lecture?

Outline - Motivation

- In battery operated devices, energy and power are very important parameters.
  - Energy and Power in PAM (chap 14)
- PAM waveforms are not WSS, thus PSD is not defined.
  - Operational Power Spectrum Density (chap 15)
- Spectral efficient and flexible modulation scheme for passband transmission.
  - Quadrature Amplitude Modulation (chap 16)

Energy in PAM

- Let's consider PAM waveform

$$X(t) = A \sum_{\ell=1}^{N} X_\ell g(t - \ell T_s)$$

  - $X_\ell \in \mathbb{R}$ are information carrying symbols (random)
  - $g \in \mathcal{L}_2$ energy-limited real pulse
  - A is a scaling factor, T_s is the baud period

$\Rightarrow$ **Note:** $X(t)$ is a stochastic process since X_ℓ are random!

Expected Energy

$$E = \mathbb{E}\left[\int_{-\infty}^{\infty} |X(t)|^2 \, \mathrm{d}t\right] = A^2 \sum_{\ell=1}^{N} \sum_{\ell'=1}^{N} \mathbb{E}\left[X_\ell X_{\ell'}\right] R_{gg}((\ell - \ell')T_s).$$

- self-similarity function $R_{gg}(\tau) = \int_{-\infty}^{\infty} g(t+\tau) g^*(t) \, \mathrm{d}t$, $\tau \in \mathbb{R}$

Discussion: Energy in PAM

- We have $E = A^2 \|g\|^2 \sum_{\ell=1}^{N} \mathbb{E}\left[X_\ell^2\right]$ if we have
  - orthogonality condition: $\int_{-\infty}^{\infty} g(t)g(t-\kappa T_s)\mathrm{d}t = \|g\|^2 \mathrm{I}\{\kappa = 0\}$, or
  - uncorrelated symbols: $\mathbb{E}[X_\ell X_{\ell'}] = \mathbb{E}\left[X_\ell^2\right] \mathrm{I}\{\ell = \ell'\}$

Binary to reals (K,N) block encoder

$$\textbf{enc} : \{0,1\}^K \to \mathbb{R}^N, \qquad D_1, \ldots, D_K \mapsto X_1, \ldots, X_N$$

- Energy per bit: $E_b \triangleq E/K$
- Energy per symbol: $E_s \triangleq E/N$
- Transmitted power: $P = E_s/T_s$
  - Hmm, does the last relation make sense (missing assumption)?

Power in PAM

Power

$$P \triangleq \lim_{T\to\infty} \frac{1}{2T} \mathbb{E}\left[\int_{-T}^{T} \|X(t)\|^2 \, \mathrm{d}t \right]$$

- If a finite number of symbols are send, then $P \to 0$ as $T \to \infty$!?!
- **Modeling trickery:** Pretend infinite sequence of symbols

$$X(t) = A \sum_{\ell=-\infty}^{\infty} X_\ell g(t - \ell T_s)$$

  - New problem: Convergence for each t?
  - Series converges if (i) symbols uniformly bounded $(X_\ell)_\ell \in \ell_\infty$ and (ii) pulse decays faster than $1/t$, i.e., $\exists \alpha, \beta > 0$: $|g(t)| \leq \frac{\beta}{1+|t/T_s|^{1+\alpha}}$.

Power in PAM if (X_ℓ) is centered WSS SP

- Centered WSS SP: $\mathbb{E}[X_\ell] = 0$ and $\mathbb{E}[X_\ell X_{\ell+m}] = K_{XX}(m)$
- Compute energy in interval $[\tau, \tau + T_s)$:

$$\mathbb{E}\left[\int_\tau^{\tau+T_s} |X(t)|^2 \, \mathrm{d}t\right] = \int_\tau^{\tau+T_s} \mathbb{E}\left[\left(A \sum_{\ell=-\infty}^{\infty} X_\ell g(t - \ell T_s)\right)^2\right] \mathrm{d}t$$

$$= A^2 \sum_{m=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \mathbb{E}[X_\ell X_{\ell+m}] \int_\tau^{\tau+T_s} g(t - \ell T_s) g(t - (\ell + m)T_s) \, \mathrm{d}t$$

$$= A^2 \sum_{m=-\infty}^{\infty} K_{XX}(m) \sum_{\ell=-\infty}^{\infty} \int_{\tau-\ell T_s}^{\tau+T_s-\ell T_s} g(t') g(t' - mT_s) \, \mathrm{d}t'$$

$$= A^2 \sum_{m=-\infty}^{\infty} K_{XX}(m) R_{gg}(mT_s), \qquad (\leftarrow \text{ does not depend on } \tau)$$

Sandwich argument

- Interval $[-T, +T)$
  - contains $\lfloor 2T/T_s \rfloor$ disjoint intervals $[\tau, \tau + T_s)$ and
  - is contained in the union of $\lceil 2T/T_s \rceil$ disjoint intervals $[\tau, \tau + T_s)$

so that

$$\left\lfloor \frac{2T}{T_s} \right\rfloor \mathbb{E}\left[\int_{\tau}^{\tau+T_s} |X(t)|^2 \,\mathrm{d}t\right] \leq \mathbb{E}\left[\int_{-T}^{T} |X(t)|^2 \,\mathrm{d}t\right] \leq \left\lceil \frac{2T}{T_s} \right\rceil \mathbb{E}\left[\int_{\tau}^{\tau+T_s} |X(t)|^2 \,\mathrm{d}t\right]$$

- **Sandwich argument:** $\lim_{T\to\infty} \frac{1}{2T} \left\lfloor \frac{2T}{T_s} \right\rfloor = \lim_{T\to\infty} \frac{1}{2T} \left\lceil \frac{2T}{T_s} \right\rceil = \frac{1}{T_s}$

$\Rightarrow$ we have $\lim_{T\to\infty} \frac{1}{2T} \mathbb{E}\left[\int_{-T}^{T} |X(t)|^2 \,\mathrm{d}t\right] = \frac{1}{T_s} \mathbb{E}\left[\int_{\tau}^{\tau+T_s} |X(t)|^2 \,\mathrm{d}t\right]$

$$P = \frac{A^2}{T_s} \sum_{m=-\infty}^{\infty} K_{XX}(m) R_{gg}(mT_s) \underset{\text{if } K_{XX}(m)=\sigma_X^2 \, \mathrm{I}\{m=0\}}{=} \frac{A^2}{T_s} \|g\|^2 \sigma_X^2$$

Time Shifts of Pulses are Orthonormal

- **Orthonormal condition:**

$$\int_{-\infty}^{\infty} \phi(t-\ell T_s)\phi(t-\ell' T_s)\,\mathrm{d}t = \mathrm{I}\{\ell = \ell'\}$$

  - Orthogonality over interval $(-\infty, \infty)$ does not to hold for $[-T, T]$
  - Require decay condition on pulses: $\exists \alpha, \beta > 0$: $|\phi(t)| \leq \frac{\beta}{1+|t/T_s|^{1+\alpha}}$.

Theorem

Consider SP $X(t) = A\sum_{\ell=-\infty}^{\infty} X_\ell \phi(t-\ell T_s)$ where $\phi(t)$ satisfies the decay and orthogonality condition and $(X_\ell) \in \ell_\infty$, then

$$\lim_{T\to\infty} \frac{1}{2T}\mathbb{E}\left[\int_{-T}^{T} |X(t)|^2\,\mathrm{d}t\right] = \frac{A^2}{T_s}\lim_{L\to\infty}\frac{1}{2L+1}\sum_{\ell=-L}^{L}\mathbb{E}\left[|X_\ell|^2\right]$$

Proof: The proof is technical and combines the previous steps.

Motivation: Operational Power Spectral Density

- **Motivation:** PSD of a WSS SP describes how the power is distributed among the frequencies.
  - PAM waveforms are not WSS!
- New concept: **Operational Power spectral density**
  - Coincides with PSD for WSS processes
  - Provides an operational meaning

- **Natural approach** following the definition of other *differential definitions of densities*, we would heuristically define the power spectral density $S_{XX}(f)$ as

$$S_{XX}(f) = \lim_{\Delta \downarrow 0} \frac{\text{Power in frequencies } [f - \frac{\Delta}{2}, f + \frac{\Delta}{2}]}{\Delta}$$

Filter Approach

- **Interpretation**: Interpret *"Power of SP $X(t)$ in frequencies $\mathcal{D}$"* as average power at the output of a filter with transfer function $\hat{h}(f) = \mathrm{I}\{f \in \mathcal{D}\}$ and SP $X(t)$ as input, i.e.,

$$\text{Power in frequencies } \mathcal{D} = \int_{-\infty}^{\infty} \mathrm{I}\{f \in \mathcal{D}\} S_{XX}(f)\,\mathrm{d}f$$

Filter Approach

Define the PSD as a function S_{XX} for which

$$\text{Power of } X \star h = \int_{-\infty}^{\infty} |\hat{h}(f)|^2 S_{XX}(f)\,\mathrm{d}f$$

holds for all BIBO stable filters , i.e., $h \in \mathcal{L}_1$ (A.L. called them "nice").

- Differential definition implies filter approach. Heuristic argument: Approximate filter $\hat{h}(f)$ by a composition of $\{0, 1\}$-filters.

Real Stochastic Processes

- We want to consider only filters with *real* impulse responses for real stochastic processes $\Rightarrow |\hat{h}(f)|^2$ is symmetric, thus

$$\int_{-\infty}^{\infty} |\hat{h}(f)|^2 S_{XX}(f)\,\mathrm{d}f = \int_{0}^{\infty} |\hat{h}(f)|^2 \Big(S_{XX}(f) + S_{XX}(-f)\Big)\,\mathrm{d}f$$

  - Only sum $S_{XX}(f) + S_{XX}(-f)$ is specified $\Rightarrow$ Non-unique

$\Rightarrow$ For sake of **uniqueness**, for real stochastic processes we additionally require $S_{XX}(f)$ to be symmetric!

  - Once one has identified a function $S(f) \geq 0$ which satisfies the integral equation, then we obtain the *symmetrized version* by

$$S_{XX}(f) = \frac{1}{2}\Big(S(f) + S(-f)\Big)$$

Operational PSD of a Real SP

Definition

The real-valued SP $X(t)$ is of **operational power spectral density** $S_{XX}(f)$ if

(i) $X(t)$ is a measurable SP;

(ii) the function $S_{XX}(f)$ is integrable and symmetric; and

(iii) for every stable real filter ($h \in \mathcal{L}_1$) the average power at the output of the filter with input $X(t)$ is given by

$$\text{Power of } X \star h = \int_{-\infty}^{\infty} |\hat{h}(f)|^2 S_{XX}(f)\,\mathrm{d}f$$

Properties

- If $\int_{-\infty}^{\infty} |\hat{h}(f)|^2 s(f)\,\mathrm{d}f = 0$ for **every complex** function $h : \mathbb{R} \to \mathbb{C}$, then $s(f)$ is zero for almost all frequencies f[1].
- If $s(f)$ is symmetric and $\int_{-\infty}^{\infty} |\hat{h}(f)|^2 s(f)\,\mathrm{d}f = 0$ holds for **every real** function $h : \mathbb{R} \to \mathbb{R}$, then $s(f)$ is zero for almost all f.

- **Uniqueness:** If S_{XX} and S'_{XX} are both operational PSD for the real SP $X(t)$, then $S_{XX}(f) = S'_{XX}(f)$ for almost all frequencies f.

Definition: Bandlimited SP

A SP $X(t)$ with operational PSD S_{XX} is **bandlimited** to W Hz, if $S_{XX} = 0$ for almost all frequencies $|f| > W$.

[1] That is, the set of frequencies at which they differ is of Lebesgue measure zero.

Operational PSD of Real PAM Signals

- Passing a pulse g through a stable filter h is equivalent to changing the pulse from g to $g \star h$.
  - Convolution is linear: $(\alpha u + \beta v) \star h = \alpha u \star h + \beta v \star h$
  - Convolution of $h(t)$ with $u(t - t_0)$ is equal to $(u \star h)(t - t_0)$

$$(X \star h)(t) = A \sum_{\ell=-\infty}^{\infty} X_\ell \, (g \star h)(t - \ell T_s)$$

⇒ Apply previous results with new pulses $g \star h$ to compute power and compare expressions!

  - E.g. (X_ℓ) centered uncorrelated with equal variance. Thus, Power of $X \star h = \frac{A^2 \sigma_X^2}{T_s} \|g \star h\|^2$

5-minute exercise

Show that $S_{XX}(f) = \frac{A^2 \sigma_X^2}{T_s} |\hat{g}(f)|^2$, $f \in \mathbb{R}$.

Further Comments

- If (X_ℓ) is centered and WSS

$$\text{Power in } X \star h = \int_{-\infty}^{\infty} \underbrace{\left(\frac{A^2}{T_s} \sum_{m=-\infty}^{\infty} K_{XX}(m) \mathrm{e}^{i2\pi f m T_s} |\hat{g}(f)|^2 \right)}_{=S_{XX}(f)} |\hat{h}(f)|^2 \mathrm{d} f$$

- Note: $S_{XX}(f)$ is a symmetric function (why?).
- About the more formal account in the textbook:
  - Issue: The convergence has to be treated more carefully.
  - Convert the problem into WSS stochastic process which requires an interesting *"stationarization argument"* and apply Wiener-Khinchin Theorem.

Let’s take a break!

Motivation Quadrature Amplitude Modulation

- Our system has **bandwidth W around carrier frequency f_c.** Thus, we can only send non-zero signals at frequencies

$$\big||f| - f_c\big| \leq W/2$$

$\Rightarrow$ We want linear modulation in passband!

- With PAM we can communicate R_s real symbols/second using pulses with bandwidth $R_s/2$ Hz $\Rightarrow$ Achievable spectral efficiency:

$$2\,\frac{\text{[real dimensions/sec]}}{\text{[baseband Hz]}}$$

- **Can this also be obtained for passband signaling? How?**

$$2\,\frac{\text{[real dimensions/sec]}}{\text{[passband Hz]}} \quad \Leftrightarrow \quad 1\,\frac{\text{[complex dimensions/sec]}}{\text{[passband Hz]}}$$

  - Simple up-conversion using $\cos(2\pi f_c t)$ doubles bandwidth!?
  - Design pulsed directly for passband is possible, but not flexible with respect to different carrier frequencies.

(One) Solution: The QAM Signal

- Easiest to describe QAM signal by looking at its **baseband representation** $x_{BB}(\cdot)$ of the transmitted passband signal $x_{PB}(\cdot)$
  - $x_{BB}(\cdot)$: Structure of PAM, but *complex symbols* C_ℓ *and pulses* $g(\cdot)$

$$X_{BB}(t) = A \sum_{\ell=1}^{n} C_\ell g(t - \ell T_s)$$

  - QAM encoder: $\phi : \{0,1\}^k \to \mathbb{C}^n$
  - Rate: k/n [bit/complex symbol]

Passband QAM signal

$$X_{PB}(t) = 2\mathrm{Re}\left(X_{BB}(t)\mathrm{e}^{i2\pi f_c t}\right) = 2\mathrm{Re}\left(A \sum_{\ell=1}^{n} C_\ell g(t - \ell T_s)\right)$$

- If g is bandlimited to $W/2$ Hz, then QAM signal is bandlimted to W Hz around carrier frequency f_c.

Pass-band signal

- ... with $\mathrm{Re}\,(wz) = \mathrm{Re}\,(w)\,\mathrm{Re}\,(z) - \mathrm{Im}\,(w)\,\mathrm{Im}\,(z)$ we get...

$$X_{PB}(t) = \sqrt{2}A\sum_{\ell=1}^{n}\mathrm{Re}\,(C_\ell)\,g_{I,\ell}(t) + \sqrt{2}A\sum_{\ell=1}^{n}\mathrm{Im}\,(C_\ell)\,g_{Q,\ell}(t)$$

- with *Inphase* and *Quadrature* components

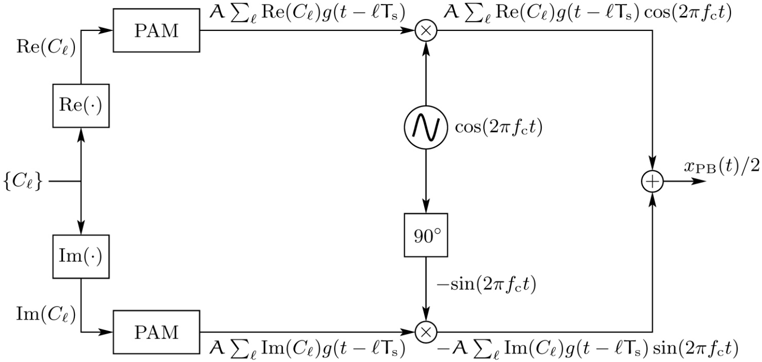
$$g_{I,\ell}(t) = 2\mathrm{Re}\left(g_{BB,I,\ell}(t)\mathrm{e}^{i2\pi f_c t}\right), \qquad g_{BB,I,\ell}(t) = \frac{1}{\sqrt{2}}g(t-\ell T_s)$$

$$g_{Q,\ell}(t) = 2\mathrm{Im}\left(g_{BB,Q,\ell}(t)\mathrm{e}^{i2\pi f_c t}\right), \qquad g_{BB,Q,\ell}(t) = i\frac{1}{\sqrt{2}}g(t-\ell T_s)$$

  - $\frac{1}{\sqrt{2}}$ factor is normalization for unit energy per dimension
  - ... if pulse g is real, then components simplify...

QAM signal generation if pulses g real

$$g_{I,\ell}(t) = \sqrt{2} g(t - \ell T_s) \cos(2\pi f_c t) \qquad g_{Q,\ell}(t) = \sqrt{2} g(t - \ell T_s) \sin(2\pi f_c t)$$



Further Comments

- **Orthonormal:** Similarly to PAM consider pulses which are orthonormal by time shifts of integer multiples of T_s

$$X_{PB}(t) = \sqrt{2}A \sum_{\ell=1}^{n} \mathrm{Re}\left(C_\ell\right) \psi_{I,\ell}(t) + \sqrt{2}A \sum_{\ell=1}^{n} \mathrm{Im}\left(C_\ell\right) \psi_{Q,\ell}(t)$$

  - Recovering via inner products (different realizations)

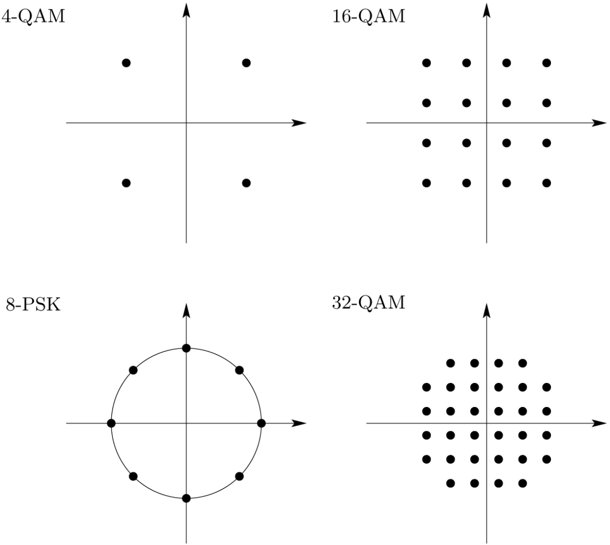
$$\mathrm{Re}\left(C_\ell\right) = \frac{1}{\sqrt{2}A} \left\langle X_{PB}, \psi_{I,\ell} \right\rangle \qquad \mathrm{Im}\left(C_\ell\right) = \frac{1}{\sqrt{2}A} \left\langle X_{PB}, \psi_{Q,\ell} \right\rangle$$

- **Spectral efficiency:** QAM with orthogonal $W/2$-bandwidth pulses $\mathrm{sinc}(Wt)$ transmits $2W$ real symbols per second and achieves spectral efficiency

$$2 \frac{\text{[real dimensions/sec]}}{\text{passband Hz}} = 1 \frac{\text{[complex dimensions/sec]}}{\text{passband Hz}}$$

- **QAM constellations $\mathcal{C}$:** 4-QAM or QPSK, M-PSK, 16 QAM,...

Some QAM constellations



Outlook - Assignment

- Energy, Power of PAM
- Operational Power Spectrum Density
- Passband Modulation: QAM

Next lecture

Complex random variables & processes; energy, power, and PSD of QAM; univariate Gaussian distribution

- Reading Assignment: Chap 17-19
- Homework:
  - Problems in textbook 14.1, 14.2, 14.3, 15.3, 15.6, 16.4, 16.5, 16.7, 16.10.
  - Deadline: Nov 7