Introduction – Stochastic Processes

Course: Foundations in Digital Communications

EQ2831 (7.5cu) Accelerated master degree course
FEO3200 (12cu) Preparatory PhD course

Tobias Oechtering
presented by: **Ragnar Thobaben**

Royal Institute of Technology (KTH),
School of EE and ACCESS Center,
Communication Theory Lab
Stockholm, Sweden

1st lecture

Outline

1. Introduction
   - Course Goals
   - Learning Activities
   - Tutorial
   - Course Pass Criterion

2. Stochastic Processes
   - Definition of continuous-time SP
   - Stationarity of SP
   - Outlook - Assignments

Course goal

Overall course goal:

Theoretical background of methods and principles used in modern digital communication systems.

Motto: Deepen existing knowledge rather than broaden.

- In particular
  - (Gaussian) stochastic processes
  - A comprehensive study of hypothesis testing
  - Concept of sufficient statistics and matched filters
  - FEO3200: Distributed detection
- Improve problem solving skills in related topics!
- FEO3200: Improve research skills!

Textbooks - Course page

- **EQ2831 & FEO3200**
  - Amos Lapidoth, “Foundations in Digital Communications,” Cambridge
  - Book on authors webpage online available
- **FEO3200**
  - Varshney, “Distributed Detection and Data Fusion,” Springer
- Course page: https://www.kth.se/social/course/EQ2831/
  - FEO3200 part is a subpage

Learning Activities

- Lectures
- Reading Assignment
- Homework: Problem solving
- Tutorial: Ticking

- Project: Report (FEO3200)

Lecture 1-8

- **EQ2831 & FEO 3200:**

#	Date	Time	Teacher	Reading assignment (chapter in textbook)
1	11/04	14-16	RT, TO	Ch. 12+13: Introduction in Stochastic Processes
2	11/06	14-16	RT	Ch. 14-16: Operational Spectral Dens., PAM, QAM
3	11/11	10-12	RT	Ch. 17-19: Energy, Power, PSD in QAM, comp.RV
4	11/18	14-16	RT	Ch. 20+21: Binary and Multi-Hypothesis Testing
5	11/25	10-12	RT	Ch. 22-24: Sufficient Statistics, Multivariate Gauss.
6	12/2	14-16	RT	Ch. 25: Continuous-Time Stochastic Process
7	12/9	9-11	TO	Ch. 26: Detection in White Gaussian Noise
8	12/16	14-16	TO	Ch. 27+28: Non-coherent Detection, Signal Detection Gaussian Noise

- Room: meeting room SIP, 3rd floor, Osquldas väg 10
- A lot of stuff to read, but concepts repeat...

Online Lectures 9-12 and Discussion Meetings

- **FEO 3200:**

#	Date	Time	Reading assignment (textbook)	Due Date
9	1/9	11-12	Ch. 2: Elements of Detection Theory	1/8
10	1/23	11-12	Ch. 3: Distrib. Detect., Parallel Fusion Network	1/22
11	2/13	11-12	Ch. 4: Distrib. Detect., Other Network Top.	2/12
12	2/27	11-12	Ch. 6+7: Distrb. Seq. Detect, IT criterion	2/26

- Room: meeting room SIP, 3rd floor, Osquldas väg 10
- Reflection assignments have to be handed in on the due date!

Homework

- Solving problems is the main learning activity.
  - Students are allowed to discuss orally only, i.e, you are not allowed to use pen and paper
  - Every student hands in its <u>own</u> solution.
  - Solutions will be checked for plagiarism!
  - **Justify each step**; if justification is missing then this is considered as wrong!
  - Every correctly solved problem is worth 1 point; partially correct gives 0.5 point; mostly wrong 0 point.
- **Strict deadline:** One day before each tutorial.
  - On paper: During office hours (before 6 PM)
  - Scanned via E-mail before midnight (send to both TAs & teachers!); one minus point for every hour late.
- Problem Set 9 for FEO3200 includes twice as many problems.

Tutorial

- Main activity: **Ticking** - students discuss HW problems.
- TA gives some hints for the next problem set.
- **TA:** Ahmed Zaki
- There will be no tutorial on homework set 8.

- **Tutorial schedule EQ2831 & FEO3200** (time: 14-16; room: SIP)

Meeting	1	2	3	4	5	6	7
Date	11/13	11/20	11/27	12/03	12/08	12/12	12/18

- **FEO3200:** Additionally tutorial Thu, 1/15, 14-16, SIP

Ticking

- In the beginning of the class every student ticks those problems for which the student believes to be able to present the solution. Meanwhile the TA discusses one problem on the next HW set.
  - Only a subset will be discussed, which problems will not be revealed.
  - For each problem the TA(!) picks student from the ticking list to present problem; the aim is to give everybody the same chances.
  - Every step has to be justified in the presentation!
  - Activity in tutorial gives you bonus points.

Tutorial Bonus Points

Participation:

- Half a bonus point if you tick more than 4 problems.

Presentation:

- Half extra bonus point if you present a mostly correct solution with reasonable justifications.
- Zero extra bonus point if the solution is wrong, but honest try.

Cheating:

- Zero bonus points for the session if you obviously do not have a solution. If this happens the second time, then all previous bonus points will be also removed!

FEO3200: Project Assignment

- **Goal:** Study a related topic and write a report.
  - Can be done in groups of two (specify contributions)
- **Form:** The project report is either
  - research paper, where new results are obtained, or
  - survey article, summary of a field in a tutorial fashion.
- **Length:** At least 5 page double column IEEE format. Good research paper might be worth to publish, then the required format of the conference or journal is sufficient.
- **Topics:** Check course page, will be updated!
- **Schedule:** Decide on a topic before last lecture, report has to be handed in before 31 March 2015!
- **TA:** Ahmed Zaki

Course Pass Criterion

All points from problem sets (unknown at the moment) = 100%.

- Student points = Problem sets points + Tutorial bonus points
- To pass course FEO3200:
  - Student points (%) $> 88\%$ of possible points from 8 problem sets
  - at least one successful (0.5p) presentations of a problem in the tutorial otherwise a successful presentation of a research paper
  - accepted report
- Grading for EQ2831:
  - at least one successful (0.5p) presentation of a problem in the tutorial otherwise a successful presentation of a research paper

Grade	A	B	C	D	E	F
Student points (%)	> 95	$95 - 88$	$88 - 77$	$77 - 63$	$63 - 50$	< 50

Let’s take a break!

Stochastic Processes

- SP play an important role in Digital Communications
  - Modeling transmitted signals
  - Modeling noise
  - Modeling other sources of impairments

Today:

- Definitions of stochastic processes (SP) (chap 12)
  - Terminology
- Properties of discrete-time stationary SP (chap 13)
  - Definition of strong and weak stationary processes
  - Autocovariance function and power spectrum density

Definitions

- **Probability space**: $(\Omega, \mathcal{F}, P)$
  - Set of experiment outcomes Ω
  - Set of events $\mathcal{F}$
  - Probability assigned of each event P (measure)

Definition: Stochastic Process (SP)

A **stochastic process** $(X(t), t \in \mathcal{T})$ is an indexed family of random variables (RVs) defined on a common probability space $(\Omega, \mathcal{F}, P)$.

- Indexing set $\mathcal{T}$
  - $\mathcal{T} = \mathbb{Z}$: *discrete-time* stochastic process $(X_\nu, \nu \in \mathbb{Z})$
  - $\mathcal{T} = \mathbb{N}$: *one-sided discrete-time* stochastic process $(X_\nu, \nu \in \mathbb{N})$
  - $\mathcal{T} = \mathbb{R}$: *continuous-time* stochastic process $(X(t), t \in \mathbb{R})$
- Aka as *random process, random function* or *random sequence*

More Definitions (terminology)

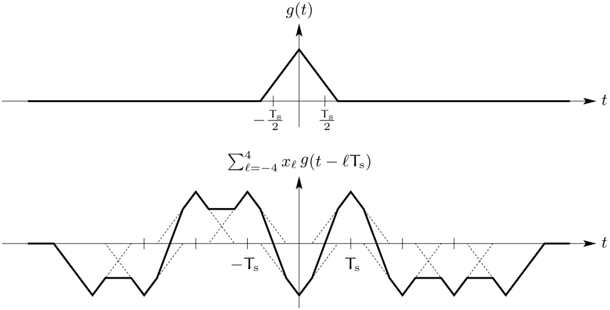
- A SP is called
  - *centered* or *of zero mean* if $\mathbb{E}[X(t)] = 0$ for all $t \in \mathcal{T}$
  - *of finite variance* if $\mathbb{E}[X^2(t)] < \infty$ for all $t \in \mathcal{T}$

Continuous-time Stochastic Process

$$X : \Omega \times \mathbb{R} \to \mathbb{R} \qquad (\omega, t) \mapsto X(\omega, t)$$

- Fix experiment outcome $\omega \in \Omega$:
  - **Function** $X(\omega, \cdot) : \mathbb{R} \to \mathbb{R}$, $t \mapsto X(\omega, t)$ is called *sample function, sample-path realization*, ...
- Fix an *epoch* $t \in \mathbb{R}$:
  - **Random variable** $X(\cdot, t) : \Omega \to \mathbb{R}$, $\omega \mapsto X(\omega, t)$ is *value of process* at time t, ...

Example: Sample path of a PAM signal



SP synthesized from (finite) sequence of RVs and pulse function g:

$$X(t) = \sum_{\ell=-4}^{4} X_\ell g(t-\ell T_s) \qquad g : t \mapsto \begin{cases} t - \frac{4}{3T_s}|t| & |t| \leq \frac{3T_s}{4} \\ 0 & |t| > \frac{3T_s}{4} \end{cases}$$

Stationary Discrete-Time Stochastic Process

Definition: Stationary Discrete-Time SP

A discrete-time SP (X_ν) is said **stationary** if

$$(X_\eta, \dots X_{\eta+n}) \overset{\mathscr{L}}{=} (X_{\eta'}, \dots X_{\eta'+n}) \qquad \forall n \in \mathbb{N}, \eta, \eta' \in \mathbb{Z}$$

- Stationarity means that consecutive random vectors have the same joint distribution (law)
- aka *strongly stationary* or *strict sense stationary*
- In particular: SP is stationary
  - $\Rightarrow$ distribution of X_ν is same as as for X_1 for all $\nu \in \mathbb{Z}$
  - $\Leftrightarrow$ $(X_{\nu_1}, \dots X_{\nu_n}) \overset{\mathscr{L}}{=} (X_{\nu_1+\eta}, \dots X_{\nu_n+\eta})$ for all $n \in \mathbb{N}$, $\nu_1, \dots \nu_n, \eta \in \mathbb{Z}$
  - $\Leftrightarrow$ $\sum_{j=1}^{n} \alpha_j X_{\nu_j} \overset{\mathscr{L}}{=} \sum_{j=1}^{n} \alpha_j X_{\nu_j+\eta}$ for all $n \in \mathbb{N}$, $\nu_j, \eta \in \mathbb{Z}$, $\alpha_j \in \mathbb{R}$, $1 \leq j \leq n$

Wide-Sense Stationary

Definition: Wide-Sense Stationary (WSS)

A discrete-time SP (X_ν) is said **wide-sense stationary** if

1) RVs X_ν are all of finite variance $\mathrm{Var}[X_\nu] < \infty$
2) RVs X_ν have identical means $\mathbb{E}[X_\nu] = \mathbb{E}[X_{\nu'}]$
3) quantity $\mathbb{E}[X_\nu X_{\nu'}]$ depends on ν and ν' only via $\nu - \nu'$

$$\mathbb{E}[X_\nu X_{\nu'}] = \mathbb{E}[X_{\nu+\eta} X_{\nu'+\eta}] \qquad \forall \eta, \nu, \nu' \in \mathbb{Z}$$

- aka *weakly stationary*
- In particular: A WSS SP

  $\Rightarrow$ $\mathrm{Var}\{X_1\} = \mathrm{Var}\{X_\nu\}, \forall \nu \in \mathbb{Z}$

  $\Leftrightarrow$ $\sum_{j=1}^{n} \alpha_j X_{\nu_j}$ and $\sum_{j=1}^{n} \alpha_j X_{\nu_j+\eta}$ have the same mean and variance for all $n \in \mathbb{N}, \nu_j, \eta \in \mathbb{Z}, \alpha_j \in \mathbb{R}, 1 \leq j \leq n$

5-minute Exercise

5 minute Exercise

Show that every finite-variance discrete-time stationary SP is WSS!

Autocovariance Function

Definition: Autocovariance Function

Let (X_ν) a WSS discrete SP, then the **autocovariance function**

$$K_{XX}(\eta) \triangleq \mathrm{Cov}[X_{\nu+\eta}X_\nu] = \mathbb{E}[X_{\nu+\eta}X_\nu] - \mathbb{E}[X_{\nu+\eta}]\mathbb{E}[X_\nu] \qquad \nu \in \mathbb{Z}$$

- In particular
  1) $K_{XX}(\eta) = \cdots = \mathbb{E}[X_{\nu+\eta}X_\nu] - \left(\mathbb{E}[X_1]\right)^2$
  2) K_{XX} is symmetric, i.e., $K_{XX}(\nu) = K_{XX}(-\nu)$
  3) $\sum_{\nu=1}^{n}\sum_{\nu'=1}^{n} \alpha_\nu \alpha_{\nu'} K_{XX}(\nu - \nu') \geq 0$ for all $\alpha_1, \ldots \alpha_n \in \mathbb{R}$
- Properties 2) and 3) denote *positive definite functions* which also characterize autocovariance functions
- Define the **autocorrelation function**

$$\rho_{XX}(\nu) \triangleq \frac{\mathrm{Cov}[X_{\nu+\eta}X_\nu]}{\mathrm{Var}[X_1]}, \qquad \nu \in \mathbb{Z}$$

Frequency domain: Power Spectral Density

Definition: Power Spectral Density (PSD)

$S_{XX}(\theta)$ is PSD of a **WSS** SP (X_ν) if Fourier coefficient of $S_{XX}(\theta)$ is

$$K_{XX}(\eta) = \int_{-1/2}^{1/2} S_{XX}(\theta) e^{-i2\pi\eta\theta} \, d\theta \qquad \forall \eta \in \mathbb{Z}.$$

- PSDs are non-negative $S_{XX}(\theta) \geq 0$ and symmetric $S_{XX}(\theta) = S_{XX}(-\theta)$ for almost all θ.
  - If PSD exists, then there exists PSD for which this holds for all θ.
  - Properties including integrable also characterize a PSD.
- If $K_{XX} \in \ell^1$ $(\sum_\eta |\cdot| < \infty)$, then $S_{XX}(\theta)$ is exists and is continuous.
- $\mathrm{Var}[X_\nu] = K_{XX}(0) = \int_{-1/2}^{1/2} S_{XX}(\theta) \, d\theta$ (power)
- If PSD does not exist, then define *spectral distribution function*.
  - Fourier series of a measure, which need not be integrable.

Outlook - Assignment

- Basic definitions, terminology, and properties of SP.
- Synthesized PAM processes are not WSS! What is the PSD?

Next lecture

Energy and Power in PAM, Operational PSD, QAM

- Reading Assignment: Chap 14-16
- Homework:
  - Problems in textbook 12.1, 12.2, 12.3, 13.1, 13.2, 13.3 (with a hint), 13.6, 13.7, 13.8
  - Deadline: Nov 12