Chapter 15 :
RAY OPTICS AND RAY MATRICES

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Aims of this chapter

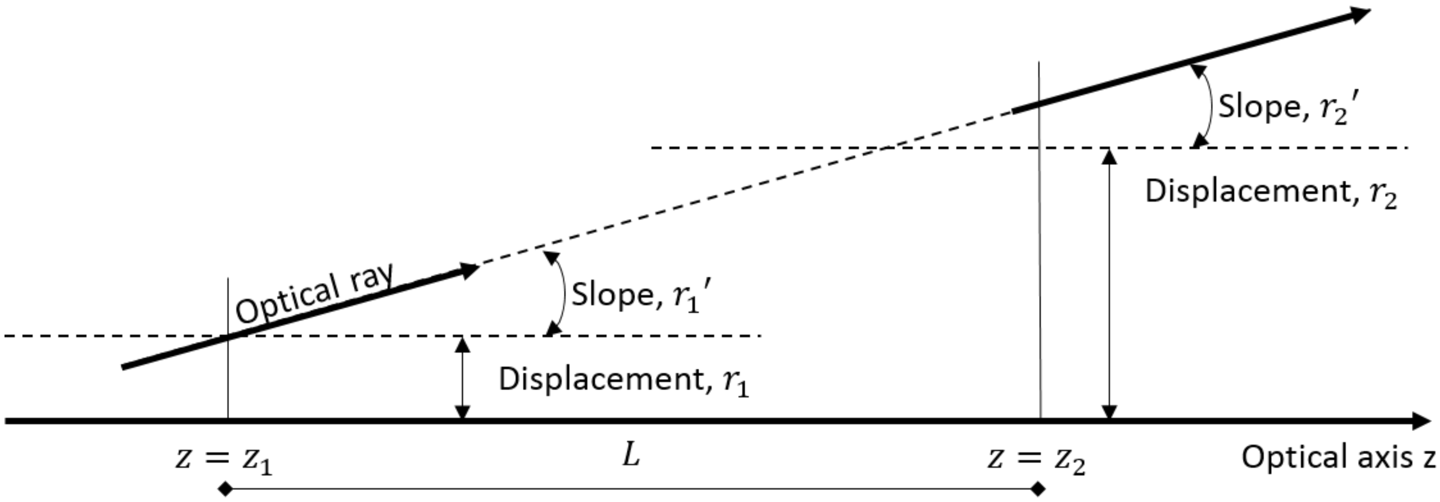
- Introduction of the matrix ABCD and its properties;
- Application of the ABCD matrix for periodic focusing systems, e.g. *optical resonator;*
- Analysis of the stability for periodic optical focusing systems;
- Evaluate the effects of misalignment of individual elements on the overall ray matrix performances;
- Introduction of the matrix ABCDEF and its properties;
- Techniques to handle the misalignment of individual elements;
- Analysis of non-orthogonal systems.

Overview

- Paraxial optical ray and ray matrices;
- Ray propagation through cascaded elements;
- Rays in periodic focusing systems;
- Ray optics with misaligned elements;
- Ray matrices in curved ducts;
- Non-orthogonal ray matrices.

Paraxial optical ray and ray matrices

OPTICAL RAY PROPAGATION IN FREE SPACE

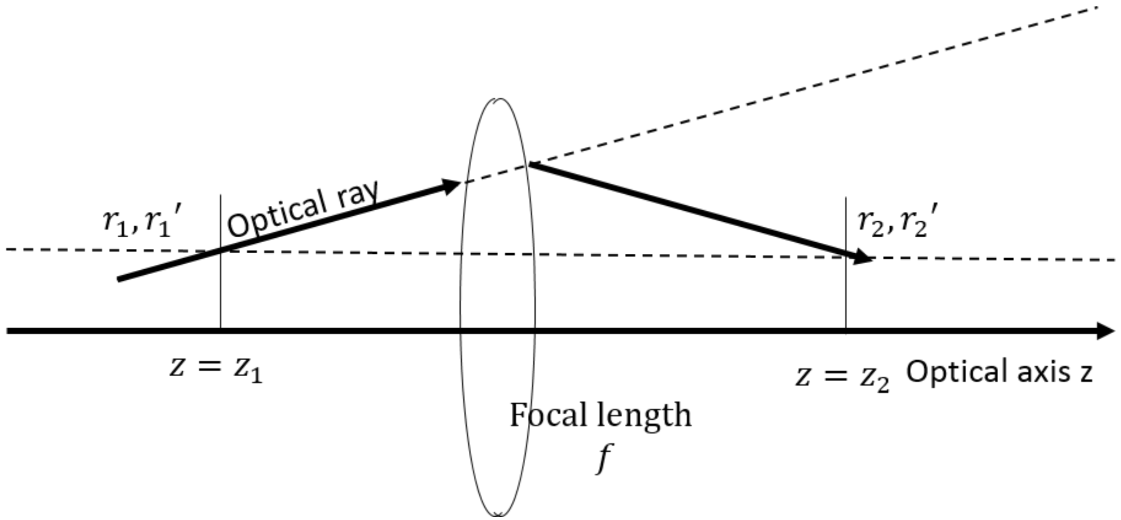


$$r_2 = r_1 + L\,\frac{dr_1}{dz}$$

$$\frac{dr_2}{dz} = \frac{dr_1}{dz}$$

Paraxial optical ray and ray matrices

OPTICAL RAY PROPAGATION THROUGH A THIN LENS

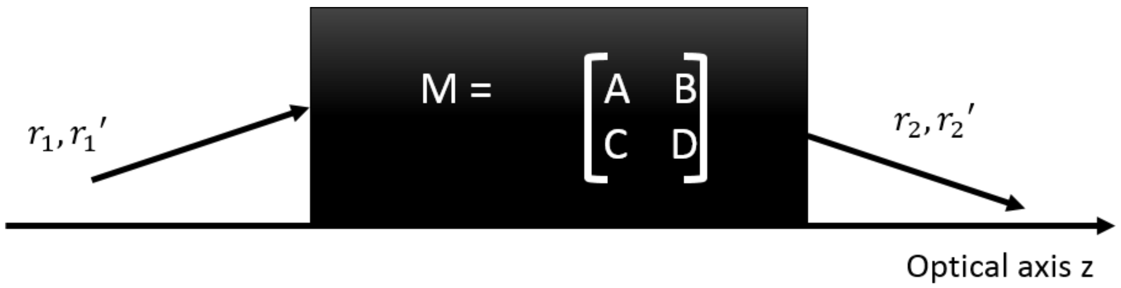


$$r_2 = r_1$$

$$\frac{dr_2}{dz} = -\frac{1}{f}r_1 + \frac{dr_1}{dz}$$

Paraxial optical ray and ray matrices

OPTICAL RAY PROPAGATION IN A GENERAL OPTICAL ELEMENT



$$r_2 = Ar_1 + Br_1'$$

$$r_2' = Cr_1 + Dr_1'$$

$$AD - BC = 1$$

With ***Reduced slope*** $r'(z) \equiv n(z)\frac{dr(z)}{dz}$

Paraxial optical ray and ray matrices

DIELECTRIC INTERFACES AND DUCTS

Paraxial System

Curved dielectric interfaces

Matrices can be derived from Snell's law and elementary geometry.

(Ref. table 15.1 LASER by Siegman)

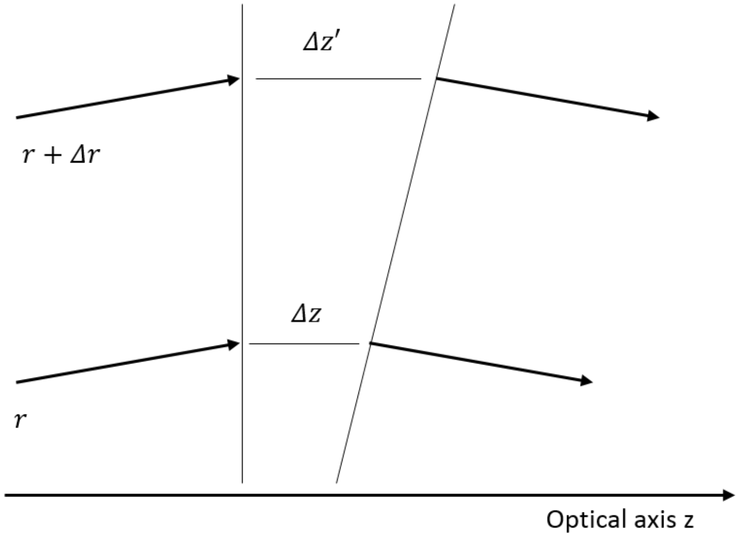
Quadratically varying dielectric media (*ducts*)

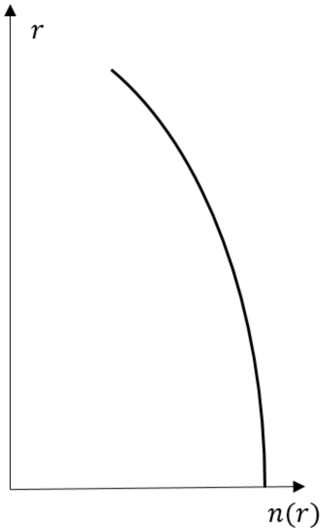
Dielectric medium which has a quadratic transverse variation in its index of reflection , with either a maximum or a minimum on the axis

Stable Vs Unstable ducts

Paraxial optical ray and ray matrices

QUADRATICALLY VARYING DIELECTRIC MEDIA





Suppose:

$$n(r,z) = n_0(z) - \frac{1}{2}n_2(z)r^2$$

With $n_2(z) = \frac{\partial^2 n(\mathrm{r,z})}{\partial z^2}$

Paraxial optical ray and ray matrices

QUADRATICALLY VARYING DIELECTRIC MEDIA

- Ray propagation equation:

$$\frac{d}{dz}\left[n_0(z)\frac{dr(z)}{dz}\right] + n_2(z)r(z) = 0$$

- But considering

$$r'(z) = \frac{dr(z)}{dz}n_0$$

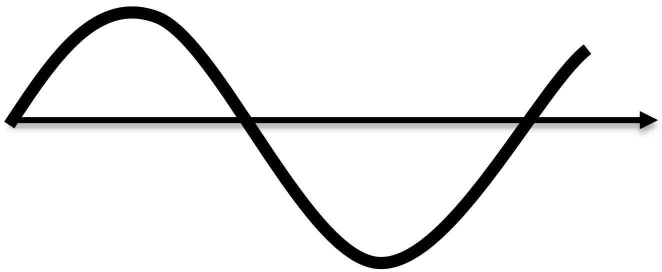
- And separating the eq.:

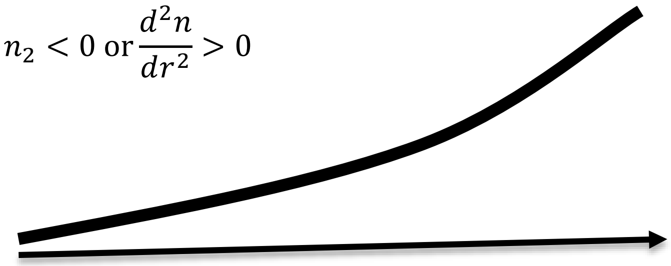
$$\frac{dr(z)}{dz} = \frac{r'(z)}{n_0(z)}$$

$$\frac{dr'(z)}{dz} = -n_2(z)\, r(z)$$

Paraxial optical ray and ray matrices

STABLE VS UNSTABLE QUADRATIC DUCTS





$$\frac{d^2r(z)}{dz^2} + \frac{n_2 r(z)}{n_0} = \frac{d^2r(z)}{dz} + \gamma^2 r(z) = 0$$

with $\gamma^2 = \frac{n_2}{n_0}$ and $\gamma = \sqrt{\frac{n_2}{n_0}}$

Solution:

$$r(z) = r_0 \cos\gamma z + (n_0\gamma)^{-1} r_0' \sin\gamma z$$

$$\begin{bmatrix} r \\ r' \end{bmatrix} = \begin{bmatrix} \cos\gamma z & (n_0\gamma)^{-1}\sin\gamma z \\ -n_0\gamma \sin\gamma z & \cos\gamma z \end{bmatrix} \begin{bmatrix} r_0 \\ r_0{}' \end{bmatrix}$$

with $\gamma^2 = -|\frac{n_2}{n_0}|$ and $\gamma = j\sqrt{\frac{1}{n_0}\frac{d^2n}{dr^2}}$

Solution:

$$r(z) = r_0 \cosh\gamma z + (n_0\gamma)^{-1} r_0' \sinh\gamma z$$

$$\begin{bmatrix} r \\ r' \end{bmatrix} = \begin{bmatrix} \cosh\gamma z & (n_0\gamma)^{-1}\sinh\gamma z \\ n_0\gamma \sinh\gamma z & \cosh\gamma z \end{bmatrix} \begin{bmatrix} r_0 \\ r_0' \end{bmatrix}$$

Paraxial optical ray and ray matrices

AXIAL INDEX VARIATIONS

- No transverse variation of the index in the medium : $n_2 = 0$;
- **Axial variation** of the index in the medium : $n_0 = n_0(z)$;
- Relevant ray equation is:

$$\frac{dr'(z)}{dz} = \frac{d}{dz}\left[n_0(z)\frac{dr}{dz}\right] = 0$$

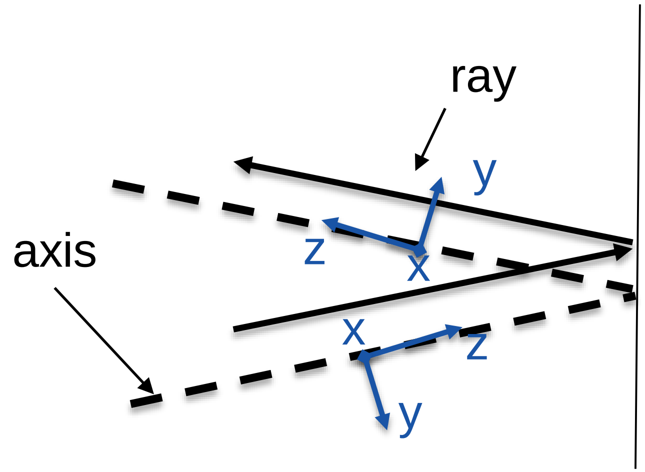
with the solution

$$r(z) = r_0 + r_0' \int_{z_0}^{z} \frac{1}{n_0(z)} dz$$

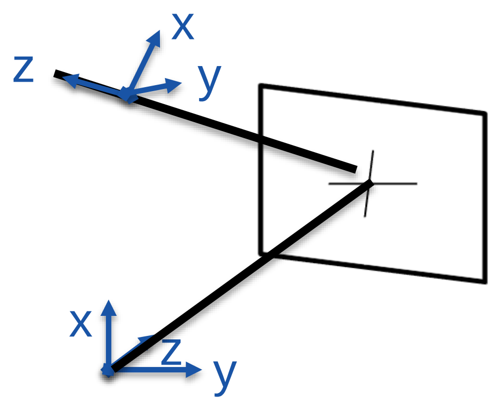
- ABCD matrix through length L starting at $z = 0$

$$M = \begin{bmatrix} 1 & B(L) \\ 0 & 1 \end{bmatrix} \text{ with } B(L) \equiv \int_0^L \frac{dz}{n_0(z)}$$

Paraxial optical ray and ray matrices



RAY INVERSION



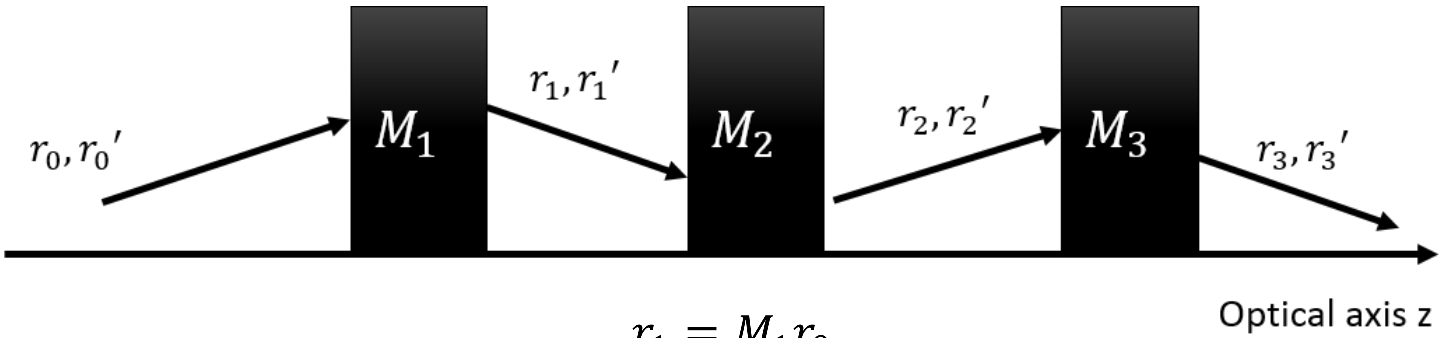
- Inversion of one optical ray with respect to one or the other of its transverse coordinate axes, e.g. mirrors, … ;
- Relation between displacement and slope before and after the reflection:

- The ray matrices along the optical axis can be written in the form:

$$\boldsymbol{x_2 = I x_1} \qquad \boldsymbol{y_2 = -I\, y_1}$$

Ray propagation through cascaded elements

CASCADE RAY MATRICES



$$r_1 = M_1 r_0$$
$$r_2 = M_2 r_1 = M_2 M_1 r_0$$
$$r_3 = M_3 r_2 = M_3 M_2 M_1 r_0$$

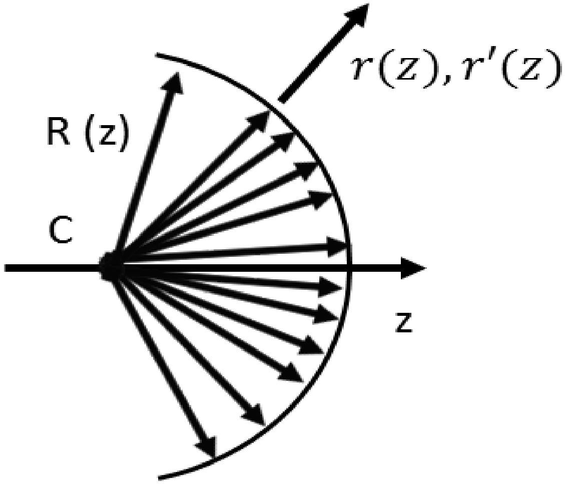
General case:

$$r_n = [M_n M_{n-1} M_{n-2} \ \ldots M_1]\, r_0 = M_{TOT} r_0$$

Ray propagation through cascaded elements

SPHERICAL WAVE PROPAGATION

- Ray optics and geometrical optics : same content expressed in different fashion;
- Ideal spherical wave with radius of curvature *R* can be seen as collection of rays diverging from common point *C*.
- Slope and displacement of each ray at the plane *z*:



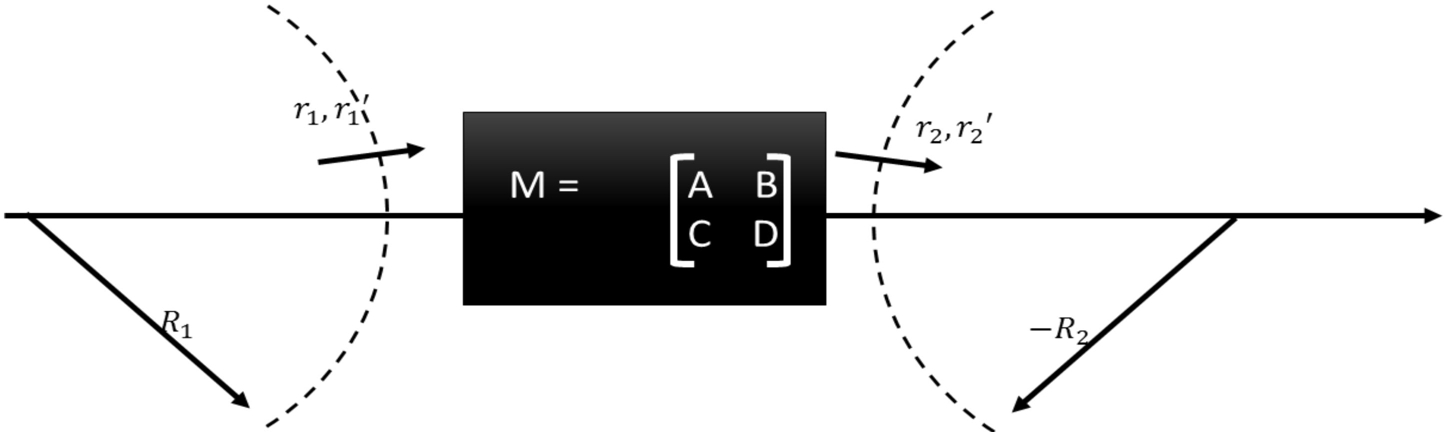
$$r'(z) = n(z)\frac{dr(z)}{dz} = n(z)\frac{r(z)}{R(z)}$$

Or

$$R(z) \equiv \frac{n(z)r(z)}{r'(z)}$$

Ray propagation through cascaded elements

SPHERICAL WAVE PROPAGATION



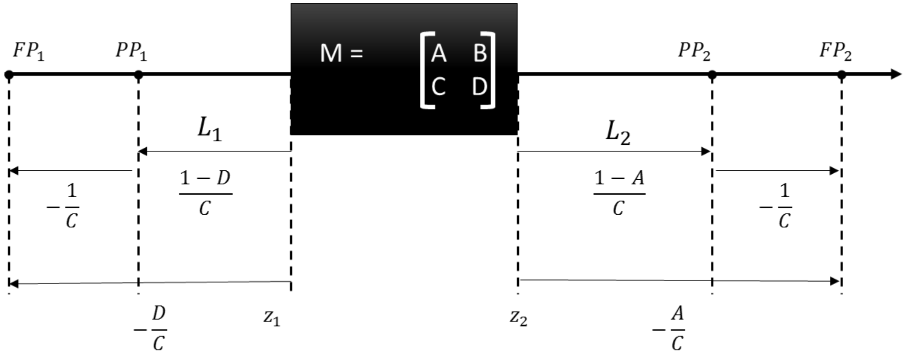
$$\frac{R_2}{n_2} \equiv \frac{r_2}{r_2'} = \frac{Ar_1 + Br_1'}{Cr_1 + Dr_1'} = \frac{A\left({R_1}/{n_1}\right) + B}{C\left({R_1}/{n_1}\right) + D}$$

With ***Reduced slope*** $\hat{R}'(z) \equiv \frac{R(z)}{n(z)}$:

$$\widehat{R_2} = \frac{A\widehat{R_1} + B}{C\widehat{R_1} + D}$$

Ray propagation through cascaded elements

THICK LENSES AND ABCD MATRICES



PP = principal plane
FP = focal plane

$$\frac{1}{\widehat{R_2} - L_2} = \frac{1}{\widehat{R_1} - L_1} + \frac{1}{1/C}$$

With $L_2 \equiv \frac{A-1}{C}$ and $L_1 \equiv \frac{1-D}{C}$

$\widehat{R_1}$ and $\widehat{R_2}$ obeys the **lens law** for a **thin lens** of focal length $f = -\frac{1}{C}$ if these quantities are measured from reference planes L_1 and L_2

Ray propagation through cascaded elements

IMAGING PROPERTIES OF ABCD SYSTEMS

- Principal plane to principal plane: $\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix}$

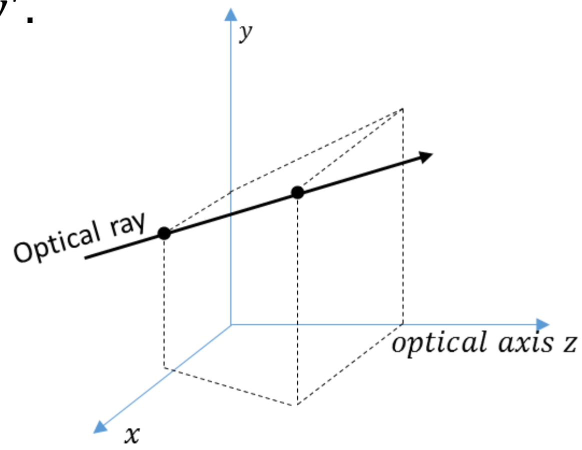
- Focal plane to focal plane: $\boldsymbol{M} = \begin{bmatrix} 0 & -C^{-1} \\ C & 0 \end{bmatrix}$

- Object plane to image plane: $\boldsymbol{M} = \begin{bmatrix} M & 0 \\ C & 1/M \end{bmatrix}$, with M image magnification

Ray propagation through cascaded elements

RAY MATRICES IN ASTIGMATIC SYSTEMS

- General ray propagating in z direction has to be described by its transverse displacement in x and y directions;
- Simple optical elements, the ray matrix formalism applies separately and independently to x, x' and y, y' ;
- If the system is rotationally symmetric the same matrix ABCD applies to both x, x' and y, y' ;
- If the system is astigmatic different matrices ABCD apply to x, x' and y, y'.



Rays in periodic focusing systems

EIGENVALUES AND EIGENRAYS

- $\boldsymbol{M}$ – matrix for propagation through one period of periodic focusing system (*PFS*) from arbitrary reference plane in one period to the corresponding plane one period later;

- $\boldsymbol{r}_n$ and $\boldsymbol{r}_{n+1}$ - ray vectors at the n-th and $n+1$-th reference planes:

$$\boldsymbol{r}_{n+1} = \boldsymbol{M}\boldsymbol{r}_n = \boldsymbol{M}^{n+1}\boldsymbol{r}_0$$

with $\boldsymbol{r}_0$ initial ray at the input plane $n = 0$;

Rays in periodic focusing systems

EIGENVALUES AND EIGENRAYS

- Look for a set of “eigenrays” $\boldsymbol{r}$ to satisfy eigenequation:

$$\boldsymbol{M}\boldsymbol{r} = \lambda\boldsymbol{r}$$

- For a 2x2 matrix :

$$[\boldsymbol{M} - \lambda\boldsymbol{I}]\boldsymbol{r} = \begin{bmatrix} A-\lambda & B \\ C & D-\lambda \end{bmatrix}\begin{bmatrix} r \\ r' \end{bmatrix} = 0$$

- Determinant must satisfy:

$$\begin{vmatrix} A-\lambda & B \\ C & D-\lambda \end{vmatrix} \equiv \lambda^2 - (A+D)\,\lambda + 1 = 0$$

Rays in periodic focusing systems

EIGENVALUES AND EIGENRAYS

- Definition of $m\ parameter$:

$$m = \frac{(\mathrm{A} + \mathrm{D})}{2}$$

- Eigenvalues:

$\lambda_a, \lambda_b = m \pm \sqrt{m^2 - 1}$, which obeys $\lambda_a \lambda_b \equiv 1$;

- Eigenrays $\boldsymbol{r_a}, \boldsymbol{r}_b$ such that:

$\boldsymbol{M}\boldsymbol{r}_a = \lambda_a \boldsymbol{r}_a$ and $\boldsymbol{M}\boldsymbol{r}_b = \lambda_b \boldsymbol{r}_b$

Rays in periodic focusing systems

EIGENRAY EXPANSION

- Arbitrary ray at the input to the PFS:

$$\boldsymbol{r}_0 = c_a \boldsymbol{r}_{\boldsymbol{a}} + c_b \boldsymbol{r}_b$$

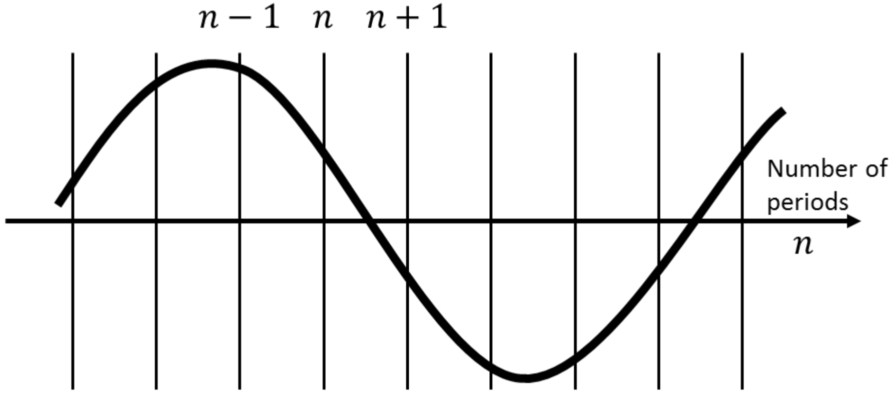
- Ray vector after any number of sections n :

$$\boldsymbol{r}_n = \boldsymbol{M}^n \boldsymbol{r}_{\boldsymbol{0}} = \boldsymbol{M}^n \times (c_a \boldsymbol{r}_{\boldsymbol{a}} + c_b \boldsymbol{r}_b)$$

$$= (c_a \times \lambda^n \boldsymbol{r}_{\boldsymbol{a}} + c_b \times \lambda^n \boldsymbol{r}_b)$$

Rays in periodic focusing systems

STABLE VS UNSTABLE PERIODIC FOCUSING SYSTEMS



$$m = \frac{A+D}{2}$$

$$|m| < 1$$

$$m \equiv \cos\theta$$

$$\lambda_a, \lambda_b = m \pm j\sqrt{1-m^2} = \cos\theta \pm j\sin\theta = e^{\pm j\theta}$$

$$\boldsymbol{r_n} = c_a\, \boldsymbol{r}_a \times e^{jn\theta} + c_b\, \boldsymbol{r}_b \times e^{-jn\theta} = \boldsymbol{r_0}\cos\theta n + \boldsymbol{s_0}\sin\theta n$$

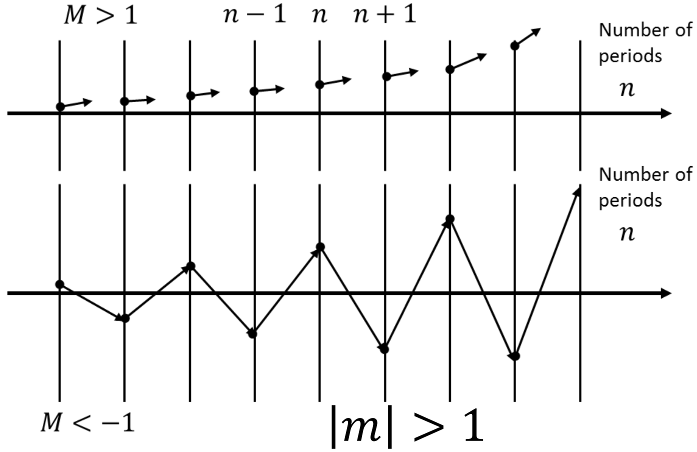
With

$$\boldsymbol{r_0} = c_a\boldsymbol{r_a} + c_b\boldsymbol{r_b}$$

$$\boldsymbol{s_0} = j(c_a\boldsymbol{r_a} + c_b\boldsymbol{r_b})$$

Rays in periodic focusing systems

STABLE VS UNSTABLE PERIODIC FOCUSING SYSTEMS



$$m = \frac{A + D}{2}$$

$$|m| > 1$$

$\lambda_a, \lambda_b = m \pm \sqrt{m^2 - 1} = M, M^{-1}$ with M *transverse magnification per period*

$$\boldsymbol{r_n} = M^n \times c_a\, \boldsymbol{r}_a + M^{-n} \times c_b\, \boldsymbol{r}_b = \boldsymbol{r_0} \cosh\theta n + \boldsymbol{s_0} sinh\theta n$$

With

$$\theta \equiv lnM$$

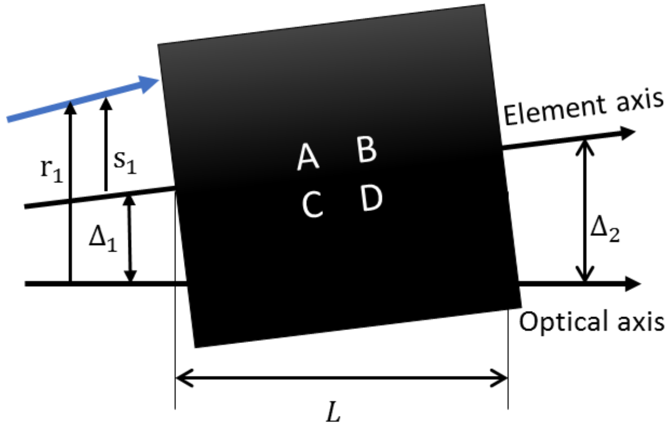
$$\boldsymbol{r_0} = c_a \boldsymbol{r_a} + c_b \boldsymbol{r_b}$$

$$\boldsymbol{s_0} = j(c_a \boldsymbol{r_a} + c_b \boldsymbol{r_b})$$

Ray optics with misaligned elements

ANALYSIS OF MISALIGNED ELEMENTS

- Real physical axis of any individual paraxial element = ***Element axis*** – ray vectors s, s';
- Reference optical axis arbitrarily chosen = ***Reference optical axis*** – ray vectors r, r';



- Considering a small angle :

$$\Delta' \equiv \frac{\Delta_2 - \Delta_1}{L}$$

Ray optics with misaligned elements

ANALYSIS OF MISALIGNED ELEMENTS

- Misalignment vectors:

$$\boldsymbol{\Delta_1} \equiv \begin{bmatrix} \Delta_1 \\ \Delta_1{}' \end{bmatrix} \text{ and } \boldsymbol{\Delta_2} \equiv \begin{bmatrix} \Delta_2 \\ \Delta_2{}' \end{bmatrix}$$

With $\Delta_1{}' \equiv n_1 \Delta'$ and $\Delta_2{}' \equiv n_2 \Delta'$reduced values

- Connection between the two vectors:

$$\boldsymbol{\Delta_2} = \begin{bmatrix} \Delta_2 \\ \Delta_2{}' \end{bmatrix} = \begin{bmatrix} 1 & L/n_1 \\ 0 & n_2/n_1 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_1{}' \end{bmatrix} \equiv \boldsymbol{M_\Delta} \times \boldsymbol{\Delta_1}$$

- Ray vectors in the element axis and the optical axis related to the input plane:

$$\boldsymbol{r_1} = \boldsymbol{s_1} + \boldsymbol{\Delta_1}$$
$$\boldsymbol{r_2} = \boldsymbol{s_2} + \boldsymbol{\Delta_2}$$

Ray optics with misaligned elements

ANALYSIS OF MISALIGNED ELEMENTS

- The ray vectors measured with respect to the element axis will transform through ABCD matrix:

$$\boldsymbol{s_2} \equiv \begin{bmatrix} s_2 \\ s_2{}' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} s_1 \\ s_1{}' \end{bmatrix} \equiv \boldsymbol{M} \times \boldsymbol{s_1}$$

With M the matrix for the aligned element

- Input and output displacements and slopes respect to the optical axis:

$$\boldsymbol{r_2} = \boldsymbol{s_2} + \boldsymbol{\Delta_2} = \boldsymbol{M s_1} + \boldsymbol{M_\Delta \Delta_1} = \boldsymbol{M r_1} + [\boldsymbol{M_\Delta} - \boldsymbol{M}]\ \boldsymbol{\Delta_1} = \boldsymbol{M r_1} + \boldsymbol{E}$$

With $\boldsymbol{E}$ ***error vector*** which is given by :

$$\boldsymbol{E} \stackrel{\text{def}}{=} \begin{bmatrix} E \\ F \end{bmatrix} = [\boldsymbol{M_\Delta} - \boldsymbol{M}]\ \boldsymbol{\Delta_1} = \begin{bmatrix} 1 - A & L - n_1 B \\ -C & n_2 - n_1 D \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta' \end{bmatrix}$$

Ray optics with misaligned elements

3X3 MATRIX FORMALISM FOR MISALIGNED ELEMENTS

Possible to put these result for a general misaligned paraxial system in a 3x3 matrix:

$$\begin{bmatrix} r_2 \\ r_2{}' \\ 1 \end{bmatrix} = \begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1{}' \\ 1 \end{bmatrix}$$

Where

$$E = (1 - A)\Delta_1 + (L \ - n_1 B)\Delta'$$

and

$$F = -C\Delta_1 + (n_2 \ - n_1 D)\Delta'$$

Ray optics with misaligned elements

CASCADE MISALIGNED ELEMENTS

- Several successive optical elements arranged in cascade where each of them is characterized by $A_i, B_i, C_i, D_i, E_i, F_i$ can be described by:

$$\left[\frac{\boldsymbol{r}_{i+1}}{1}\right] = \left[\frac{\boldsymbol{M}}{\boldsymbol{O}}\middle|\frac{\boldsymbol{E}}{1}\right]\left[\frac{\boldsymbol{r}_i}{1}\right]$$

With M is the ABCD matrix, $\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{E}$ are 2x1 column matrices, $\boldsymbol{O}$ is a 2x1 row matrix with all elements 0 and 1 is 1x1 element.

- Example of two misaligned elements:

$$\left[\frac{\boldsymbol{M}_{tot}}{\boldsymbol{O}}\middle|\frac{\boldsymbol{E}_{tot}}{1}\right] = \left[\frac{\boldsymbol{M}_2}{\boldsymbol{O}}\middle|\frac{\boldsymbol{E}_2}{1}\right] \times \left[\frac{\boldsymbol{M}_1}{\boldsymbol{O}}\middle|\frac{\boldsymbol{E}_1}{1}\right] = \left[\frac{\boldsymbol{M}_2\boldsymbol{M}_1}{\boldsymbol{O}}\middle|\frac{\boldsymbol{M}_2\boldsymbol{E}_1 + \boldsymbol{E}_2}{1}\right]$$

Ray optics with misaligned elements

CASCADE MISALIGNED ELEMENTS

The basic ray matrix properties and paraxial focusing properties of a cascade system are entirely unchanged by small misalignments of individual elements within the system.

Ray optics with misaligned elements

OVERALL MISALIGNED SYSTEMS

- Initial ray $\boldsymbol{r_0}, N$ elements, misalignment described by $\boldsymbol{E_k} \equiv [E_k, F_k]$;
- The overall transformation is:

$$\boldsymbol{r_N} = \boldsymbol{M}_{TOT}\boldsymbol{r}_0 + \boldsymbol{E}_{TOT}$$

Where

$$\boldsymbol{M}_{TOT} = \boldsymbol{M}_N \ldots \boldsymbol{M_2}\boldsymbol{M_1}$$

and

$$\boldsymbol{E}_{TOT} = [\boldsymbol{M}_N \ldots \boldsymbol{M_2}]\boldsymbol{E_1} + [\boldsymbol{M}_N \ldots \boldsymbol{M_3}]\boldsymbol{E_2} + \ldots + \boldsymbol{M}_N\, \boldsymbol{E_{N-1}} + \boldsymbol{E_N}$$

Ray optics with misaligned elements

SYSTEM ALIGNMENT AND OVERALL ELEMENT AXIS

- Any system can be converted into an effectively aligned overall system by bringing its *overall element axis* into coincidence with the *system's element axis*;
- Any overall values $E = E_{TOT}$ and $F = F_{TOT}$ for an overall system can be cancelled out:
  - Physical *translation* of entire system as a unit downward:

$$\Delta_0 = \frac{(1 - \mathrm{D})\mathrm{E} - (\mathrm{L} - \mathrm{B})\mathrm{F}}{(1 - \mathrm{A})(1 - \mathrm{D}) + (\mathrm{L} - \mathrm{B})\mathrm{C}}$$

  - Physical *rotation* of entire system toward system axis, with the centre of rotation at the input plane, by an angle:

$$\Delta' = \frac{\mathrm{CE} + (1 - A)\mathrm{F}}{(1 - \mathrm{A})(1 - \mathrm{D}) + (\mathrm{L} - \mathrm{B})\mathrm{C}}$$

Ray optics with misaligned elements

MISALIGNED RESONATOR OR PERIODIC SYSTEMS

- Consider an unfold optical resonator with one or more misaligned internal elements;
- Each round trip in the resonator (or individual period of the lensguide) will have an overall element axis, with respect to which that individual round trip will look like an aligned system;
- This element axis will not come back on itself after one round trip: the element axis of each may be tilted to the reference optical axis running through the repeated section of the lensguide, so that element axes in successive periods do not connect;
- "*Is there any better or alternative way to define an effective axis in a misaligned resonator or periodic system?*"

Ray optics with misaligned elements

MISALIGNED RESONATOR OR PERIODIC SYSTEMS

- Aligned paraxial systems require: a ray vector which starts perfectly aligned along the axis remains always aligned;
- Define this “unique” axis ray vector: $\boldsymbol{r_0}$;
- Conditions necessary for $\boldsymbol{r_0}$ for self-reproducing after a round trip:

$$\boldsymbol{M}\boldsymbol{r_0} + \boldsymbol{E} = \boldsymbol{r_0} \quad \text{or} \quad \boldsymbol{r_0} = (\boldsymbol{I} - \boldsymbol{M})^{-1}\boldsymbol{E}$$

- Displacement and slope of the axis ray vector:

$$r_0 \equiv \frac{(1-D)E + BF}{2 - A - D}\,, \qquad r_0' \equiv \frac{CE + (1-A)F}{2 - A - D}$$

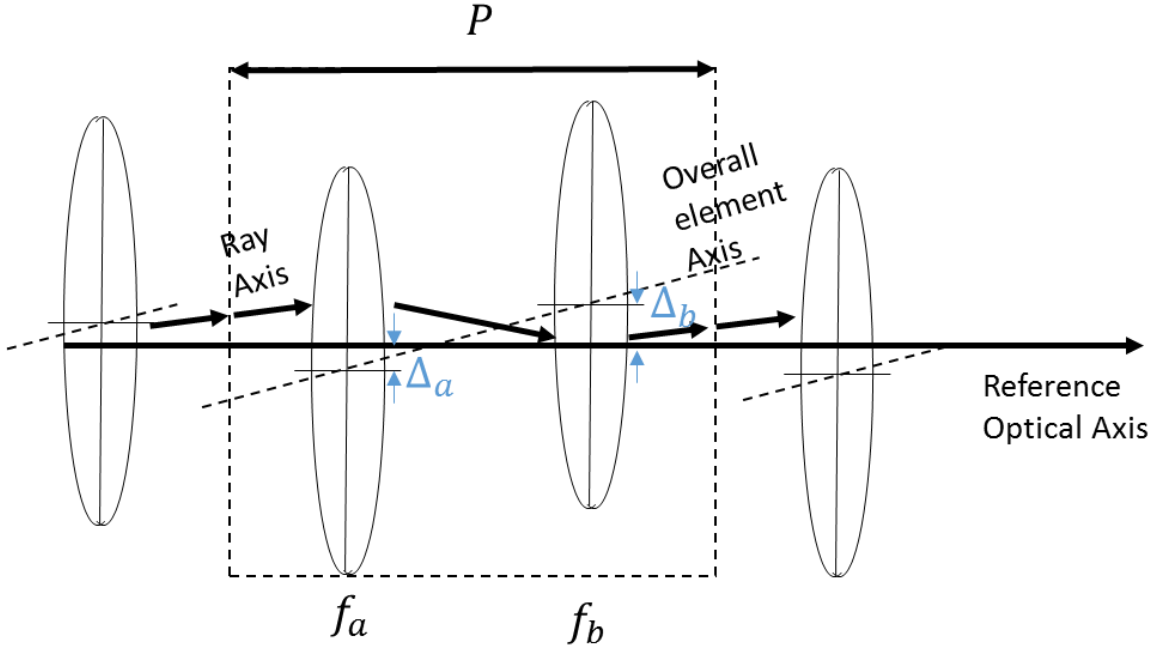
- Transformation of any other input ray $\boldsymbol{r_1}$ through the misaligned system is given by:

$$(\boldsymbol{r_2} - \boldsymbol{r_0}) = \boldsymbol{M} \times (\boldsymbol{r_1} - \boldsymbol{r_0})$$

Ray optics with misaligned elements

DIFFERENCES BETWEEN THE AXIS RAY AND THE OVERALL ELEMENT AXIS

Axis Ray	Overall Element Axis
• Bent curves and segments; • Parallel to itself after one pass through the system.	• Straight line through the elements.



Ray matrices in curved ducts

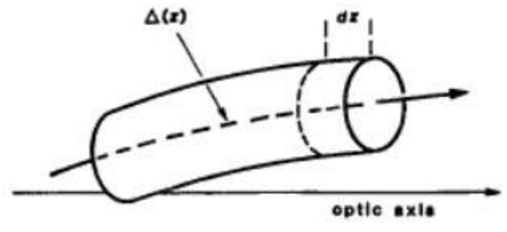
DIFFERENTIAL MATRIX ANALYSIS

- Consider a quadratic ducts:
  - $\Delta(z)$ displacement of the axis of the duct at any plane z from the optical axis;
  - $\boldsymbol{M}(z)$ 3x3 ABCDEF matrix from an input plane z_0 up to a plane z;
  - $\boldsymbol{M}(dz)$ 3x3 ABCDEF matrix for a short distance dz from z to $\mathrm{z} + dz$:

$$\boldsymbol{M}(dz) = \begin{bmatrix} 1 & n_0^{-1} dz & 0 \\ -n_0 \gamma^2 dz & 1 & n_0 \gamma^2 \Delta(z) dz \\ 0 & 0 & 1 \end{bmatrix} \text{ for } dz \to 0$$

  - For the cascading properties of the ray matrix :

$$\boldsymbol{M}(z + dz) = \boldsymbol{M}(z) \times \boldsymbol{M}\,(dz)$$



Ray matrices in curved ducts

DIFFERENTIAL MATRIX ANALYSIS

- Multiplying $\boldsymbol{M}(dz)$ and $\boldsymbol{M}(z)$ and comparing with the matrix $\boldsymbol{M}(z+dz)$, differential relations are given:

$$\frac{dA(z)}{dz} = n_0^{-1}C(z), \qquad \frac{dB(z)}{dz} = n_0^{-1}D(z)$$

$$\frac{dC(z)}{dz} = -n_0\gamma^2 A(z), \qquad \frac{dD(z)}{dz} = -n_0\gamma^2 B(z)$$

$$\frac{dE(z)}{dz} = -n_0^{-1}F(z), \qquad \frac{dF(z)}{dz} = -n_0\gamma^2[E(z) - \Delta(z)]$$

- Solving the first four equations, starting from z_0 gives the overall ABCD matrix as a function of the distance:

$$A(z) = D(z) = cos\gamma(z - z_0)$$

$$n_0\gamma B(z) = -(n_0\gamma)^{-1}C(z) = \sin\gamma(z - z_0)$$

- The overall matrix is unchanged by curvature of duct.

Ray matrices in curved ducts

EFFECT OF DUCTS MISALIGNMENT

$$\frac{dE(z)}{dz} = -n_0^{-1}F(z)$$

$$\frac{dF(z)}{dz} = -n_0\gamma^2[E(z) - \Delta(z)]$$

- Formal solutions:

$$E(z) = \gamma \int_{z_0}^{z} \Delta(z') \sin\gamma\,(z - z')dz'$$

$$F(z) = n_0\gamma^2 \int_{z_0}^{z} \Delta(z') \cos\gamma\,(z - z')dz'$$

Ray matrices in curved ducts

EFFECT OF DUCTS MISALIGNMENT

$$E(z) = \gamma \int_{z_0}^{z} \Delta(z') \sin\gamma\,(z - z')dz'$$

$$F(z) = n_0\gamma^2 \int_{z_0}^{z} \Delta(z') \cos\gamma\,(z - z')dz'$$

- Case 1: $\Delta(z) = \cos\gamma_1 z$ or $\Delta(z) = \sin\gamma_1 z$, with γ_1 closely matches the natural ray oscillations at $cos\gamma z$ or $sin\gamma z$:
    - Displacement parameters $E(z), F(z)$ will grow linearly with the distance,
    - The periodic oscillations of rays in the duct will appear to grow linearly in amplitude with the distance;
- Case 2: $\Delta(z)$ has random variations :
    - Oscillations in off-axis rays will grow as the square root of distance along the guide,
    - The growth rate will be proportional to the amplitude of the spatial frequency components of $\Delta(z)$ in the vicinity of γ.

Non-orthogonal Ray Matrices

GENERAL ANLYSIS

- *Non-orthogonal* systems:
  - Systems which can not be described by separate and independent ray matrices in two principal plane that are 90° apart,
  - Exhibit some kind of "twist" or image rotation;
- Notation for a non-orthogonal 4x4 matrix:

$$\begin{bmatrix} x_2 \\ y_2 \\ \hline x_2' \\ y_2' \end{bmatrix} = \left[\begin{array}{cc|cc} A_{xx} & A_{xy} & B_{xx} & B_{xy} \\ A_{yx} & A_{yy} & B_{yx} & B_{yy} \\ C_{xx} & C_{xy} & D_{xx} & D_{xy} \\ C_{yx} & C_{yy} & D_{yx} & D_{yy} \end{array}\right] \begin{bmatrix} x_1 \\ y_1 \\ \hline x_1' \\ y_1' \end{bmatrix}$$

- Or in short notation

$$\left[\frac{\boldsymbol{r}_2}{\boldsymbol{r}_2'}\right] = \left[\frac{\boldsymbol{A}}{\boldsymbol{C}}\middle|\frac{\boldsymbol{B}}{\boldsymbol{D}}\right] \times \left[\frac{\boldsymbol{r}_1}{\boldsymbol{r}_1'}\right]$$

Non-orthogonal Ray Matrices

GENERAL ANLYSIS

$$\begin{bmatrix} x_2 \\ y_2 \\ \hline x'_2 \\ y'_2 \end{bmatrix} = \left[\begin{array}{cc|cc} A_{xx} & A_{xy} & B_{xx} & B_{xy} \\ A_{yx} & A_{yy} & B_{yx} & B_{yy} \\ C_{xx} & C_{xy} & D_{xx} & D_{xy} \\ C_{yx} & C_{yy} & D_{yx} & D_{yy} \end{array}\right] \begin{bmatrix} x_1 \\ y_1 \\ \hline x'_1 \\ y'_1 \end{bmatrix}$$

- Constraints:

$$\boldsymbol{AB^T = BA^T}$$
$$\boldsymbol{B^T D = D^T B}$$
$$\boldsymbol{DC^T = CD^T}$$
$$\boldsymbol{C^T A = A^T C}$$
$$\boldsymbol{AD^T - BC^T = A^T D - B^T C = I}$$

Non-orthogonal Ray Matrices

- Alternative form:

$$\begin{bmatrix} x_2 \\ x'_2 \\ \hline y_2 \\ y'_2 \end{bmatrix} = \left[\begin{array}{cc|cc} A_{xx} & B_{xx} & A_{xy} & B_{xy} \\ C_{xx} & D_{xx} & C_{xy} & D_{xy} \\ A_{yx} & B_{yx} & A_{yy} & B_{yy} \\ C_{yx} & D_{yx} & C_{yy} & D_{yy} \end{array}\right] \begin{bmatrix} x_1 \\ x'_1 \\ \hline y_1 \\ y'_1 \end{bmatrix}$$

- Or in short notation

$$\begin{bmatrix} \boldsymbol{x}_2 \\ \hline \boldsymbol{y}_2 \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{M}_{xx} & \boldsymbol{M}_{xy} \\ \hline \boldsymbol{M}_{yx} & \boldsymbol{M}_{yy} \end{array}\right] \times \begin{bmatrix} \boldsymbol{x}_1 \\ \hline \boldsymbol{y}_1 \end{bmatrix}$$

Non-orthogonal Ray Matrices

ROTATED ASTIGMATIC OPTICAL SYSTEMS

- *Coordinate system rotation* :
  - Matrix for an orthogonal system:

$$\begin{bmatrix} x_2 \\ x'_2 \\ \hline y_2 \\ y'_2 \end{bmatrix} = \left[\begin{array}{cc|cc} A_{xx} & B_{xx} & 0 & 0 \\ C_{xx} & D_{xx} & 0 & 0 \\ 0 & 0 & A_{yy} & B_{yy} \\ 0 & 0 & C_{yy} & D_{yy} \end{array}\right] \begin{bmatrix} x_1 \\ x'_1 \\ \hline y_1 \\ y'_1 \end{bmatrix}$$

  - At any z, coordinate rotation from original x_1, y_1 to x_2, y_2:

$$\begin{bmatrix} x_2 \\ x'_2 \\ \hline y_2 \\ y'_2 \end{bmatrix} = \left[\begin{array}{cc|cc} cos\theta & 0 & sin\theta & 0 \\ 0 & cos\theta & 0 & sin\theta \\ -sin\theta & 0 & cos\theta & 0 \\ 0 & sin\theta & 0 & cos\theta \end{array}\right] \begin{bmatrix} x_1 \\ x'_1 \\ \hline y_1 \\ y'_1 \end{bmatrix}$$

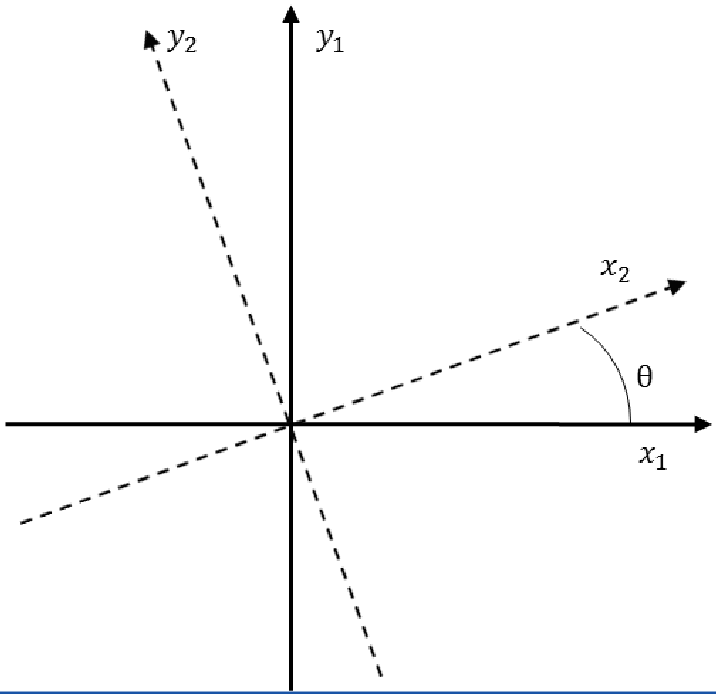
Where subscript 1 refers to the old coordinate system and subscript 2 refers to the rotated coordinate system.

Non-orthogonal Ray Matrices

ROTATED ASTIGMATIC OPTICAL SYSTEMS

- Short notation:

$$\left[\frac{\boldsymbol{x_2}}{\boldsymbol{y_2}}\right] = \left[\begin{array}{c|c} \dfrac{C_\theta}{-S_\theta} & \dfrac{S_\theta}{C_\theta} \end{array}\right] \times \left[\frac{\boldsymbol{x_1}}{\boldsymbol{y_1}}\right]$$



Non-orthogonal Ray Matrices

ROTATED ASTIGMATIC OPTICAL SYSTEMS

- To pass a ray through this rotated element analytically using the original x_1, y_1 axes:
  - Transform from the original axes into rotated principal axes of the element,
  - Propagate through the element using ABCD matrix along its principal axes,
  - Rotate back to original axes by amount $-\theta$;

$$\begin{bmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{bmatrix} \times \begin{bmatrix} \boldsymbol{M}_{xx} & 0 \\ 0 & \boldsymbol{M}_{yy} \end{bmatrix} \times \begin{bmatrix} C_\theta & S_\theta \\ -S_\theta & C_\theta \end{bmatrix}$$

Which can be manipulated in the form:

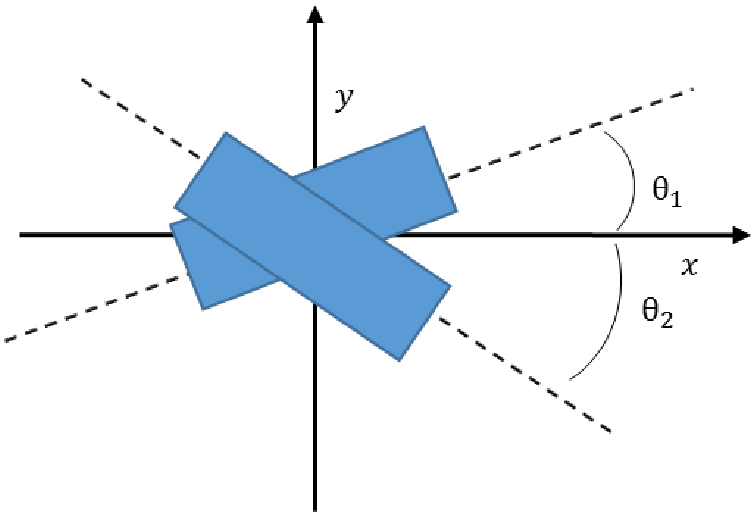
$$\begin{bmatrix} C_\theta^2 \boldsymbol{M}_{xx} + S_\theta^2 \boldsymbol{M}_{yy} & C_\theta S_\theta (\boldsymbol{M}_{xx} - \boldsymbol{M}_{yy}) \\ C_\theta S_\theta (\boldsymbol{M}_{xx} - \boldsymbol{M}_{yy}) & S_\theta^2 \boldsymbol{M}_{xx} + C_\theta^2 \boldsymbol{M}_{yy} \end{bmatrix}$$

Non-orthogonal Ray Matrices

TWO ROTATED ELEMENTS IN CASCADE

- Two orthogonal but astigmatic elements in cascade, rotated respectively of θ_1, θ_2;
- Overall matrix product:

$$\left[\begin{array}{c|c} \boldsymbol{M_{xx}} & \boldsymbol{M_{xy}} \\ \hline \boldsymbol{M_{yx}} & \boldsymbol{M_{yy}} \end{array}\right] = [overall\ 4 \times 4\ matrix\ product]$$



Non-orthogonal Ray Matrices

TWO ROTATED ELEMENTS IN CASCADE

- After some calculation it is possible to demonstrate:

$$M_{xy} - M_{yx} = \sin(\theta_2 - \theta_1)\, cos(\theta_2 - \theta_1)(\boldsymbol{M}_{xx,1} - \boldsymbol{M}_{yy,1})(\boldsymbol{M}_{xx,\mathbf{2}} - \boldsymbol{M}_{yy,\mathbf{2}})$$

- A cascade system of two rotated astigmatic elements can be orthogonal only if:
  - $\theta_2 - \theta_1 = 0°$, so the principal planes of the two elements coincide,
  - $\boldsymbol{M}_{xx,1} = \boldsymbol{M}_{yy,1}$ or $\boldsymbol{M}_{xx,\mathbf{2}} = \boldsymbol{M}_{yy,\mathbf{2}}$, so one of the elements is not astigmatic;

- *"An optical system having cascaded astigmatic elements rotated at arbitrary angles will in general not be orthogonal".*

Non-orthogonal Ray Matrices

- *Image rotation:*
  - Can be describe with the same matrix used for the coordinate system rotation

$$\begin{bmatrix} x_2 \\ x'_2 \\ \hline y_2 \\ y'_2 \end{bmatrix} = \left[\begin{array}{cc|cc} cos\theta & 0 & sin\theta & 0 \\ 0 & cos\theta & 0 & sin\theta \\ -sin\theta & 0 & cos\theta & 0 \\ 0 & sin\theta & 0 & cos\theta \end{array}\right] \begin{bmatrix} x_1 \\ x'_1 \\ \hline y_1 \\ y'_1 \end{bmatrix}$$

Non-orthogonal Ray Matrices

NON PLANAR RING RESONATOR

- Coordinates rotation Vs. image rotation very difficult to distinguish for a twisted or non planar ring resonator;
- Can be described by:

$$\left[\begin{array}{c|c} \dfrac{C_\theta \boldsymbol{M}_{xx}}{-S_\theta \boldsymbol{M}_{yy}} & \dfrac{S_\theta \boldsymbol{M}_{xx}}{C_\theta \boldsymbol{M}_{yy}} \end{array}\right]$$

If the rotation element comes first or by

$$\left[\begin{array}{c|c} \dfrac{C_\theta \boldsymbol{M}_{xx}}{-S_\theta \boldsymbol{M}_{xx}} & \dfrac{S_\theta \boldsymbol{M}_{yy}}{C_\theta \boldsymbol{M}_{yy}} \end{array}\right]$$

If the astigmatic element comes first.

Assignments

TASK 1

Problems for 15.1 – Ex. 1 - page 592 of the book “Laser” by Siegman

Suggestions:

- Use small angles approximation;
- Consider the thickness of the interface $t = 0$ m.

Assignments

TASK 2

Consider an optical resonator of length L made by two intracavity lenses of focal length $f\ = 2L$ equally spaced between two flat end mirrors.

1. The system is **aligned**. Find the general ABCD matrix of the system for any round trip in the resonator.

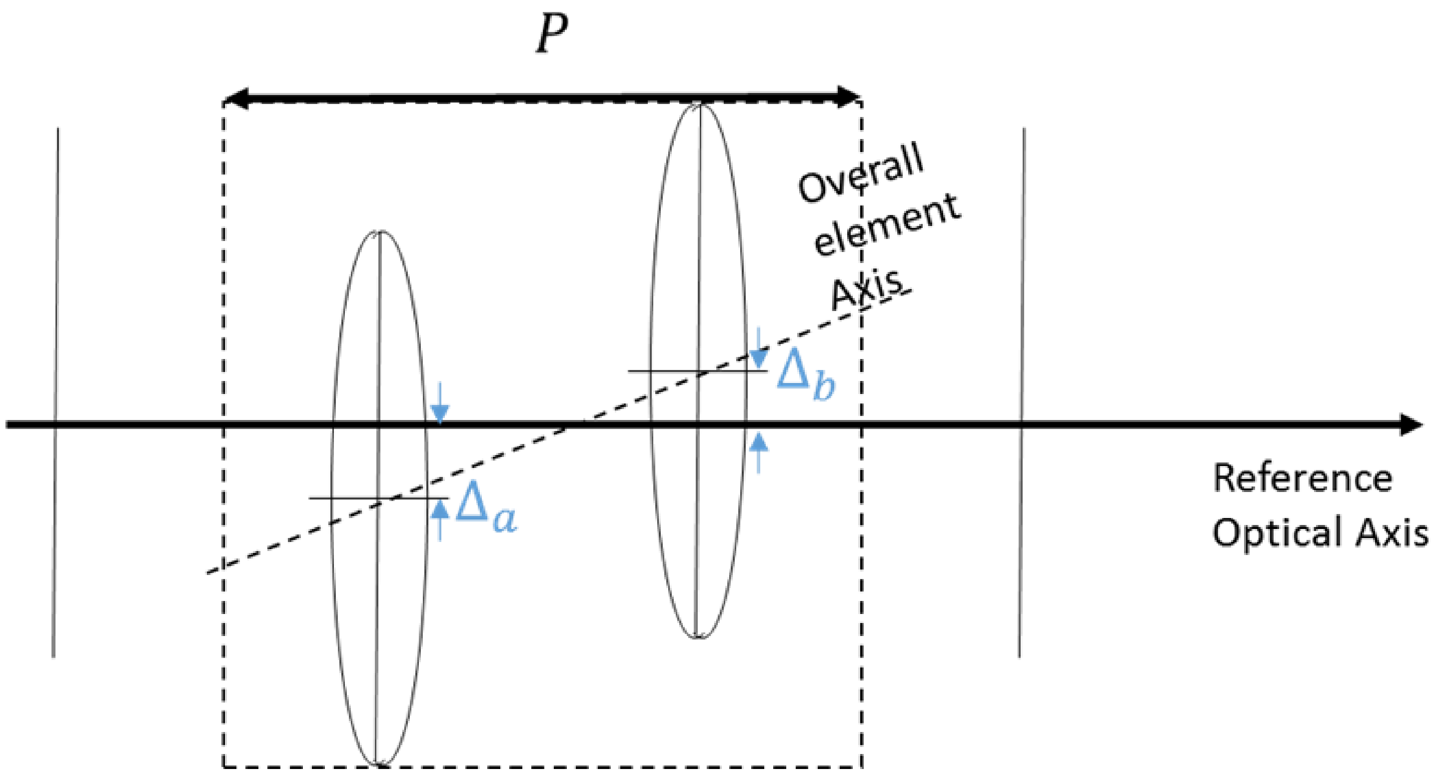
2. The system is **misaligned**. The first lens is displaced below the optical axis of a distance $\Delta_a =\ 2\varepsilon$ and the second lens is displaced upward of a distance $\Delta_b = \varepsilon$. Calculate the overall element axis passing through the two lenses. Consider the two lenses grouped in a single element of dimension $P =\ 2/3L$, cantered in $L/2$. Calculate the misalignment of the element respect the reference axis Δ' and considering the element described by a general matrix ABCD, write the general equations for E and F.

Assignments

TASK 2

Suggestion:

- Consider the thin lenses approximation.



Thank you for the attention!