Lecture 5
Optical resonators*

Max Yan
Photonics, KTH

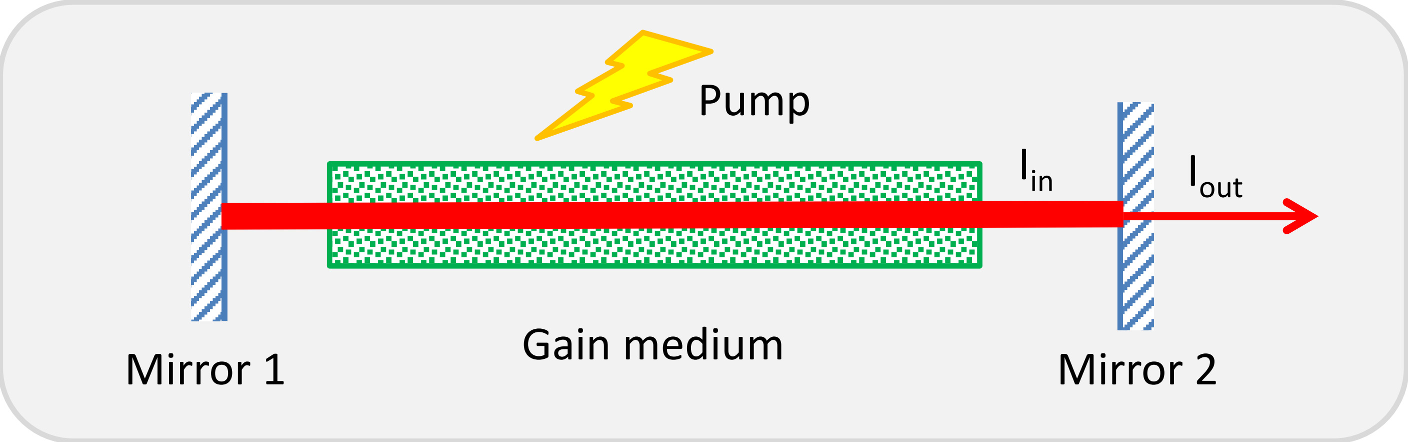
* Some figures and texts belong to: O. Svelto, *Principles of Lasers*, 5th Ed., Springer.

Reading

- *Principles of Lasers* (5th Ed.): Chapter 5.
- Skip: 5.5.2, 5.5.3, 5.6.2.
- Squeeze*: 5.5.1, 5.6.

* "Squeeze" means to get basic physical feeling of the section.

Laser



Cavity field: $E(x, y, z, t) = \tilde{E}(x, y, z) \exp\left(j\omega t - \frac{t}{2\tau_c}\right)$

not yet active

Quality: stability, photon lifetime

General properties:
- Partially open
- Long (cm to <1m)
- Transverse and longitudinal modes

Contents

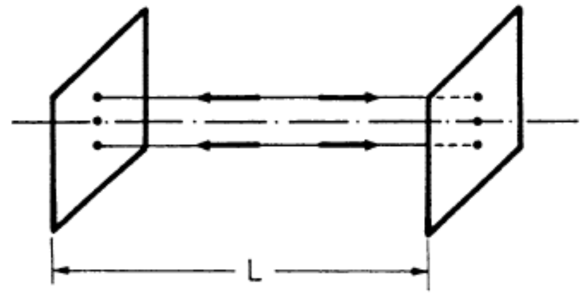
Contents

Content	Time
1. Resonator varieties	15′
2. Cavity Q	10′
3. Stability condition	15′
4. Wave-optics mode solutions	35′
5. Unstable resonators	5′
Total:	**80′**

KTH VETENS

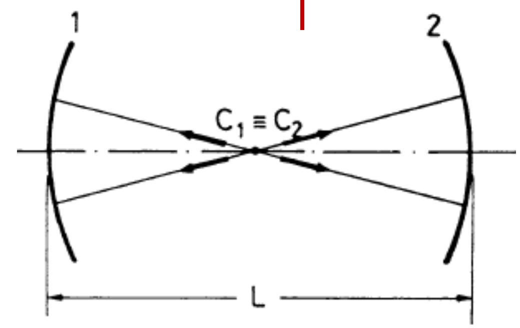
Resonator varieties

- Plane-parallel (FP)

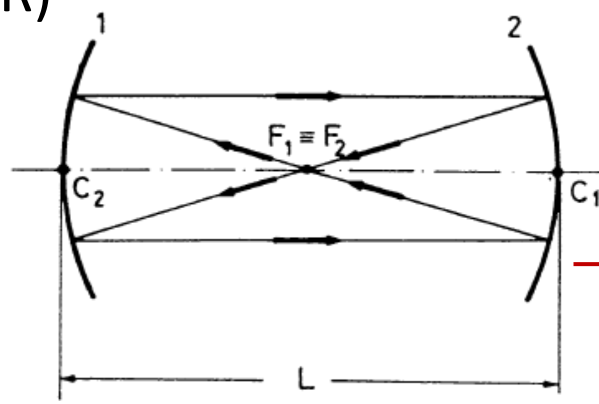


$$\nu_n = n\frac{c}{2L}$$

- Concentric (L=2R)

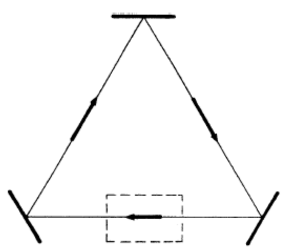


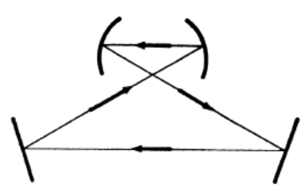
- Confocal (L=R)



$$\nu_n = n\frac{c}{4L}$$

- Ring



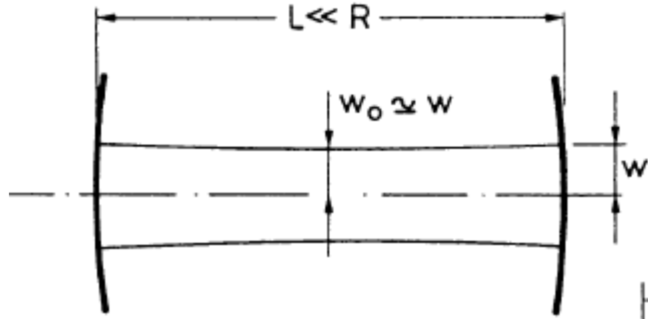


Others: R<L<2R; different Rs

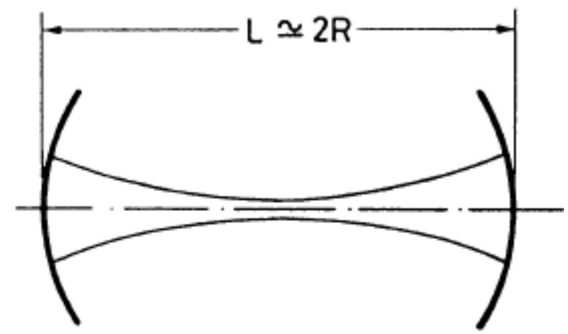
- ○ Stable
- ○ Unstable

Resonator varieties

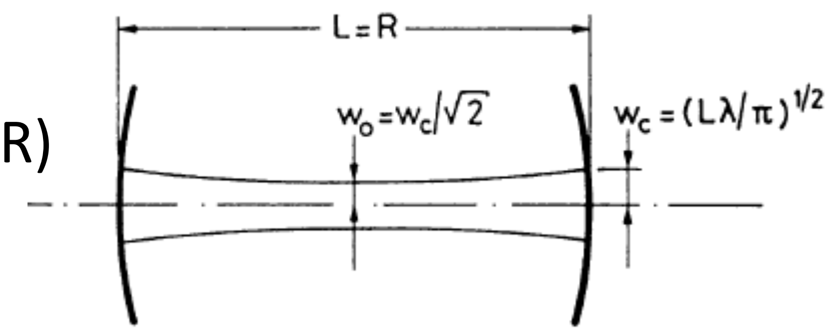
- Plane-parallel (FP)



- Concentric (L=2R)



- Confocal (L=R)



ν_n=?
TEM_{lm}?

TEM_{00}

$$\tilde{E}(x,y,z) \propto \exp\left[-jk\left(z+\frac{x^2+y^2}{2q}\right)\right]$$

$$\frac{1}{q} = \frac{1}{R} - j\frac{\lambda}{\pi w^2}$$

Radius of curvature

Beam spot size

Contents

Content	Time
1. Resonator varieties	15′
2. Cavity Q	10′
3. Stability condition	15′
4. Wave-optics mode solutions	35′
5. Unstable resonators	5′
Total:	**80′**

Photon lifetime

Light intensity after *m* round-trips: $I(t_m) = [R_1 R_2 (1 - T_i)^2]^m I_0$

Photons after *m* round-trips: $\phi(t_m) = \left[R_1\, R_2\, (1 - T_i)^2\right]^m\, \phi_0$

Phenomenological assumption: $\phi(t) = \exp\left(-\frac{t}{\tau_c}\right) \phi_0$ $\qquad t = m\frac{2L}{c}$

$$\tau_c = -\frac{2L}{c \ln\left[R_1 R_2 (1 - T_i)^2\right]} = \frac{L}{c\gamma}$$

γ: single-pass logrithmic cavity loss

Time-dependence of field: $E(t) = \tilde{E} \exp\left(-\frac{t}{2\tau_c} + j\omega t\right)$

By Fourier transform: $\Delta\nu_c = \frac{1}{2\pi\tau_c}$

FWHM of resonance peak (MHz range)

Quality factor

$$Q = 2\pi \frac{\text{energy stored in cavity}}{\text{energy lost in one cycle of oscillation}}$$

$$\begin{aligned} Q &= 2\pi \frac{h\nu\phi}{h\nu\left(-\frac{d\phi}{dt}\right)\frac{1}{\nu}} \\ &= 2\pi\nu\tau_c \\ &= \frac{\nu}{\Delta\nu_c} \end{aligned}$$

Example

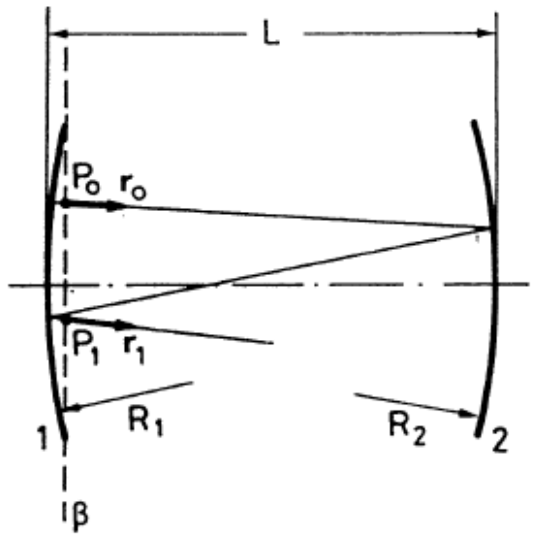
$R_1 = R_2 = 0.98$, $T_i = 0$, L=90cm, $\lambda = 630$nm

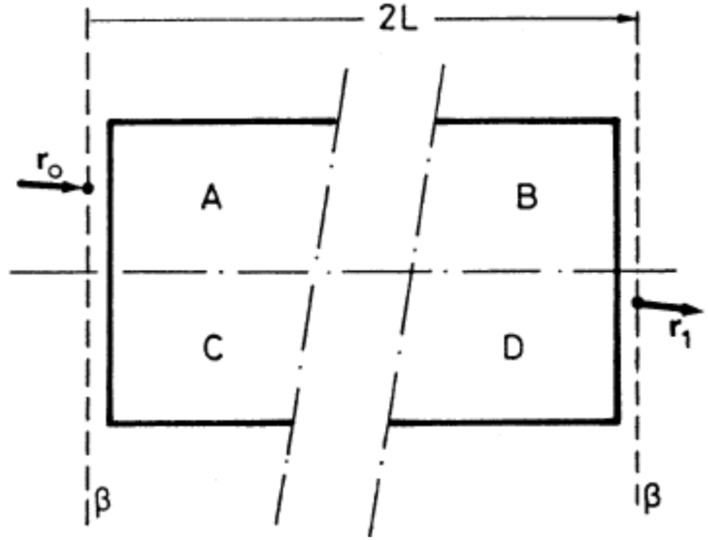
→ $\Delta\upsilon_c = 1.1$MHz, $Q = 4.7\times10^8$.

Contents

Stability condition

Ray optics



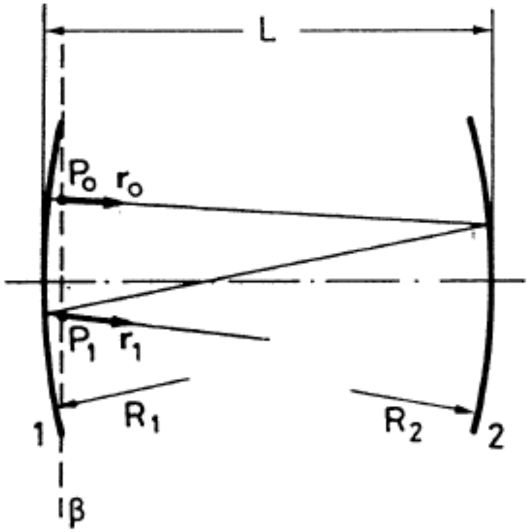


After one roundtrip $\begin{vmatrix} r_1 \\ r_1' \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \begin{vmatrix} r_0 \\ r_0' \end{vmatrix}$

After n roundtrips $\begin{vmatrix} r_n \\ r_n' \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}^n \begin{vmatrix} r_0 \\ r_0' \end{vmatrix}$

For overall ABCD matrix not to diverge $-1 < \left(\frac{A+D}{2}\right) < 1$

Two-spherical-mirror case Ray optics

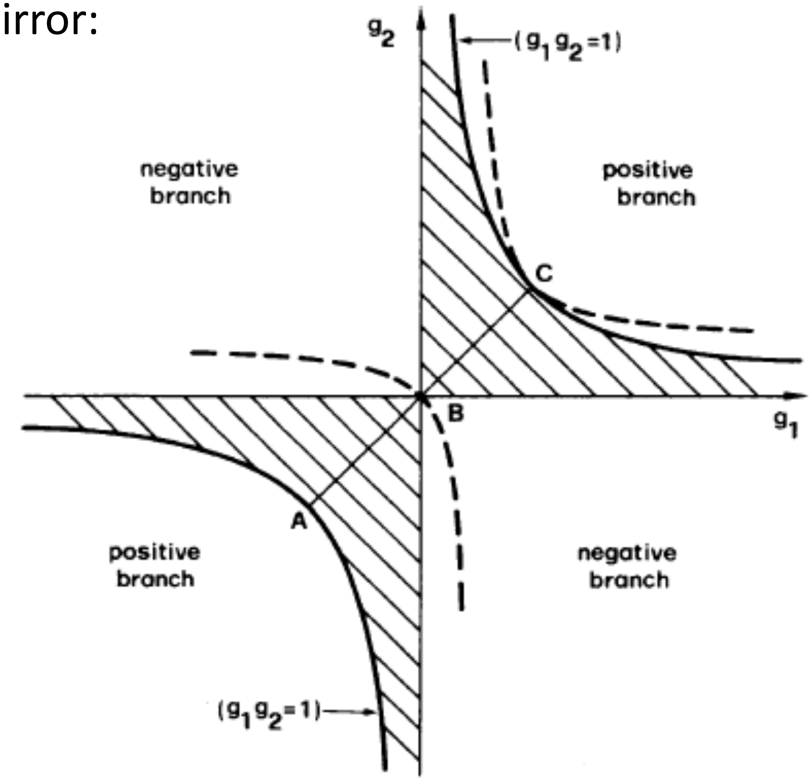


ABCD matrix
for spherical mirror:

$$\begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix}$$

for free space:

$$\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$



After finding round-trip ABCD matrix,

$$\frac{A+D}{2} = 2\left(1-\frac{L}{R_1}\right)\left(1-\frac{L}{R_2}\right)-1$$

If one defines:

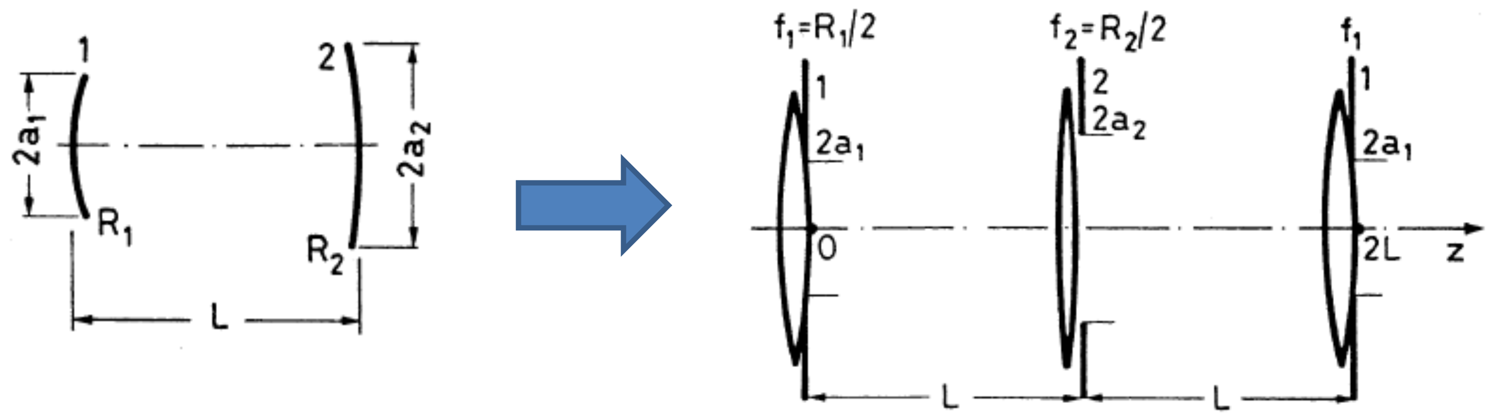
$$g_1 = 1-(L/R_1)$$
$$g_2 = 1-(L/R_2)$$

Stability condition: $0 < g_1 g_2 < 1$

A: concentric
B: confocal
C: planar

Contents

Wave optics

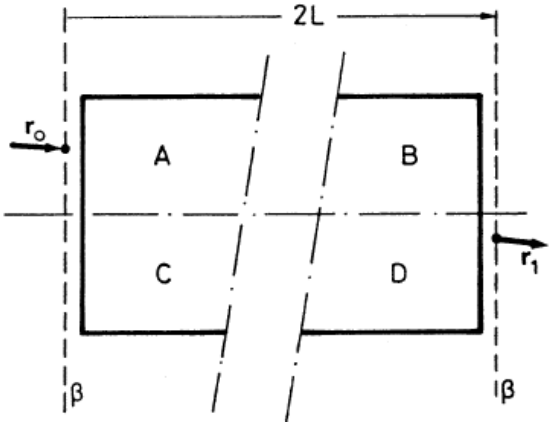


$$\tilde{E}(x,y,2L) =$$

$$\exp(-2jkL)\iint_{\infty}\frac{j}{B\lambda}\exp\left[-jk\frac{A(x_1^2+y_1^2)+D(x^2+y^2)-2x_1x-2y_1y}{2B}\right]\tilde{E}(x_1,y_1,0)dx_1dy_1$$

Fresnel-Kirchoff integral

Stability condition



Modes: TEM transverse modes

ABCD law of Gaussian beam propagation

$$q = \frac{A\,q_1 + B}{C\,q_1 + D}$$

q must repeat itself after a round trip:

$$q = \frac{Aq + B}{Cq + D} \longrightarrow \boxed{Cq^2 + (D - A)q - B = 0}$$

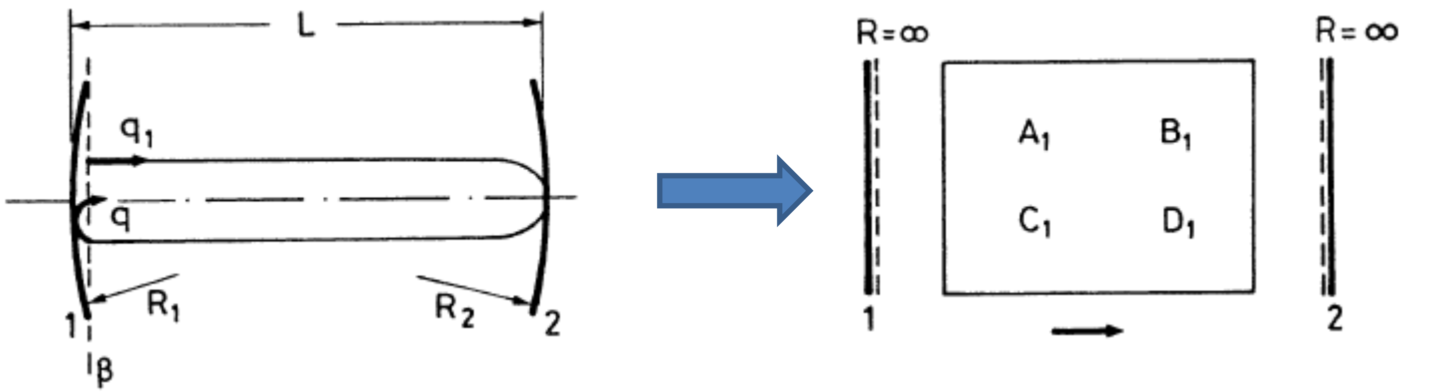
q must be complex, hence $(D - A)^2 + 4BC < 0$

Since AD-BC=1 $(D + A)^2 < 4$

Stability condition: $-1 < \left(\frac{A + D}{2}\right) < 1$ (re-derived)

Mode properties

Two-mirror case



Round-trip matrix: $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 2A_1D_1 - 1 & 2B_1D_1 \\ 2A_1C_1 & 2A_1D_1 - 1 \end{vmatrix}$

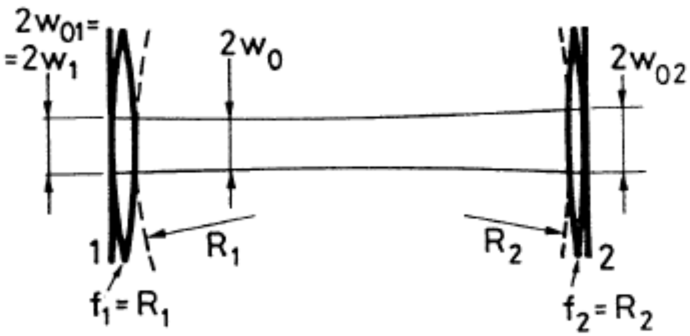
Since A=D, from red eq. in previous slide $q = q_1 = j\sqrt{-\frac{B}{C}} = j\sqrt{-\frac{B_1D_1}{A_1C_1}}$

Implication: q_1 is purely imaginary ≡ equiphase surface coincides with mirror 1

Implication 2: beam spot size can be computed based on q_1.

$$\frac{1}{q} = \frac{1}{R} - j\frac{\lambda}{\pi w^2}$$

Beam size



Step 1: $$\begin{vmatrix} A_1 & B_1 \\ C_1 & D_1 \end{vmatrix} = \begin{vmatrix} g_1 & L \\ -(1-g_1g_2)/L & g_2 \end{vmatrix}$$

Step 2: $$q = q_1 = j\sqrt{-\frac{B}{C}} = j\sqrt{-\frac{B_1D_1}{A_1C_1}}$$

Step 3: $$w_1 = \left(\frac{L\lambda}{\pi}\right)^{1/2}\left[\frac{g_2}{g_1\ (1-g_1g_2)}\right]^{1/4}$$

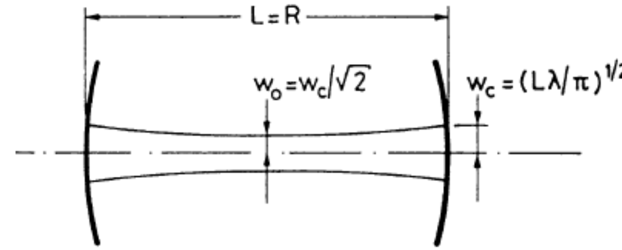
Special case: symmetric mirrors (R_1=R_2=R)

$$w = \left(\frac{L\lambda}{\pi}\right)^{1/2}\left[\frac{1}{1-g^2}\right]^{1/4}$$

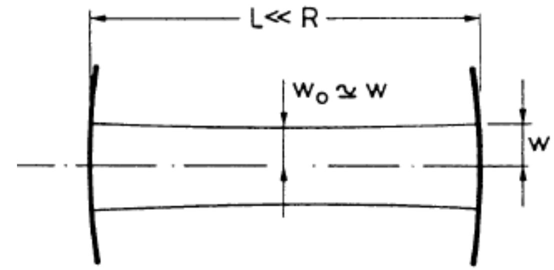
$$w_0 = \left(\frac{L\lambda}{\pi}\right)^{1/2}\left[\frac{1+g}{4\,(1-g)}\right]^{1/4}$$

Beam size: special cases

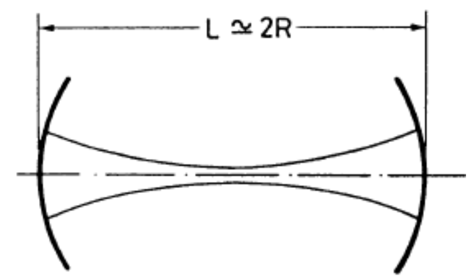
•Confocal (g=0): $w_c = \sqrt{\frac{L\lambda}{\pi}},\quad w_{0c} = \frac{w_c}{\sqrt{2}}$



•Near-plane (g=1-ε, ε=L/R small): $w_c\sqrt[4]{\frac{1}{2\epsilon}},\quad w_c\sqrt[4]{\frac{1}{2\epsilon}}$



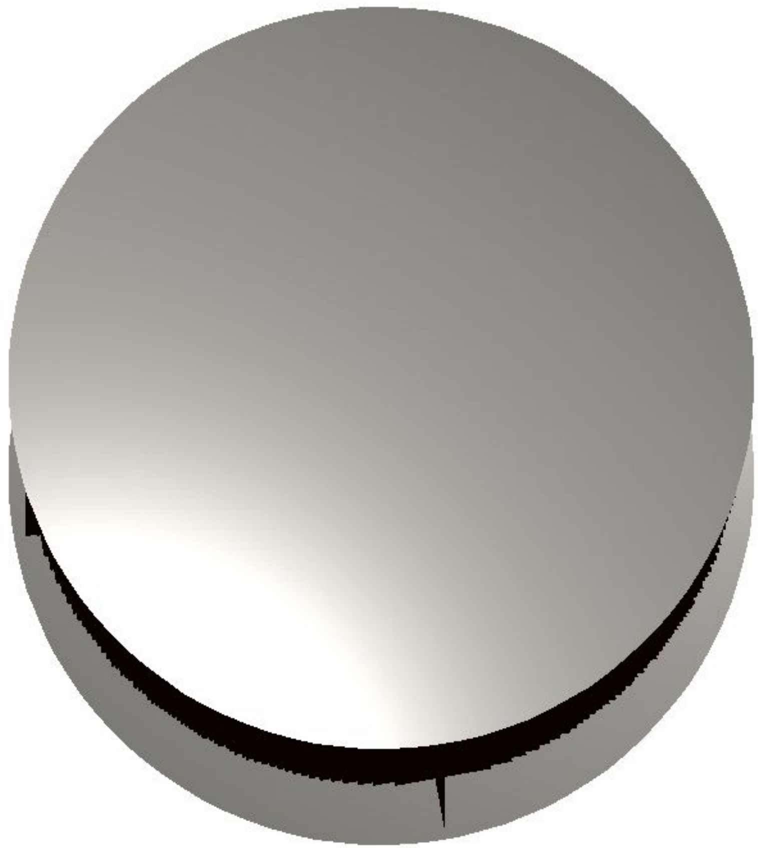
•Concentric (g=-1+ ε, ε small): $w_c\sqrt[4]{\frac{1}{2\epsilon}},\quad w_c\sqrt[4]{\frac{\epsilon}{8}}$



Example:
L=1m, λ=514nm,
•Confocal: w_c=0.4mm; w_0=0.29mm
•Near-plane (R=10m): w_c=0.61mm; w_0=0.59mm
•Concentric (R=0.5m, ε=0.1): w_c=0.61mm; w_0=0.14mm

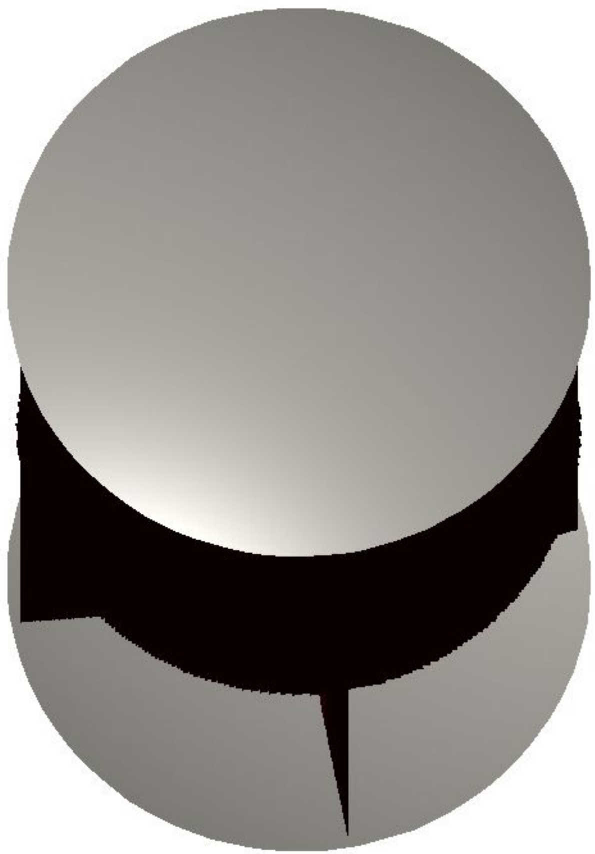
A TEM$_{00}$ cavity mode

A TEM_{00} cavity mode

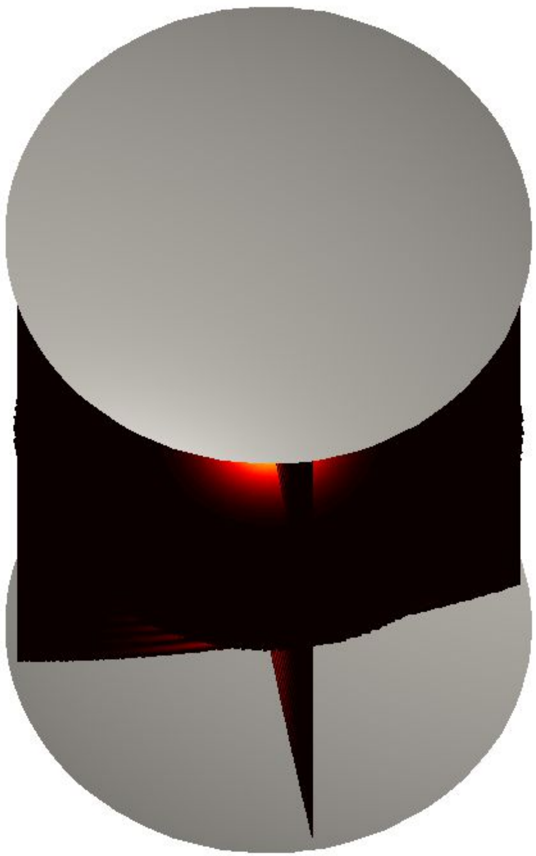


A TEM_{00} cavity mode

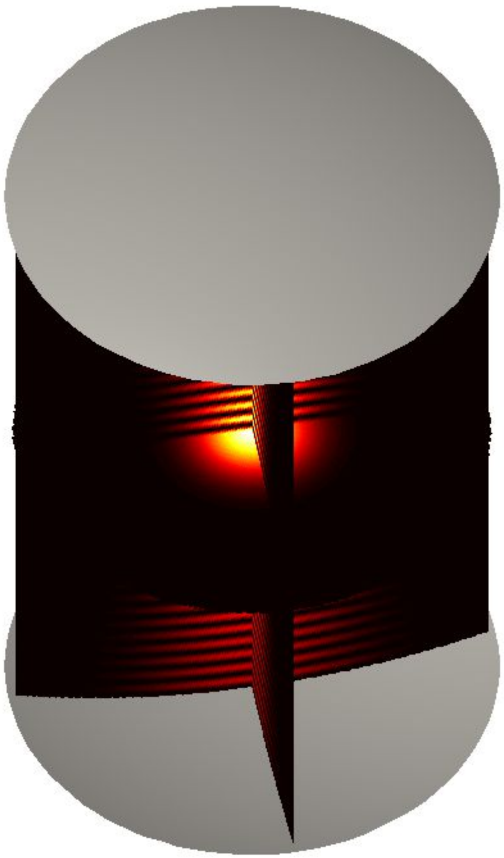
A TEM_{00} cavity mode



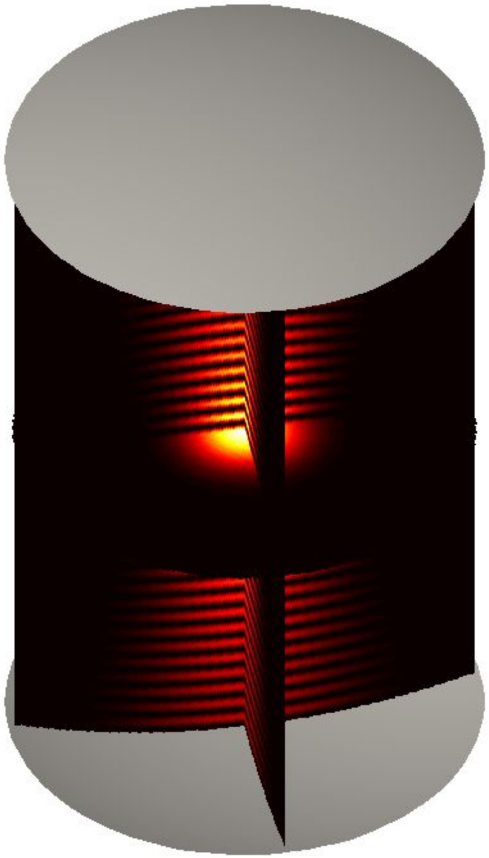
A TEM_{00} cavity mode



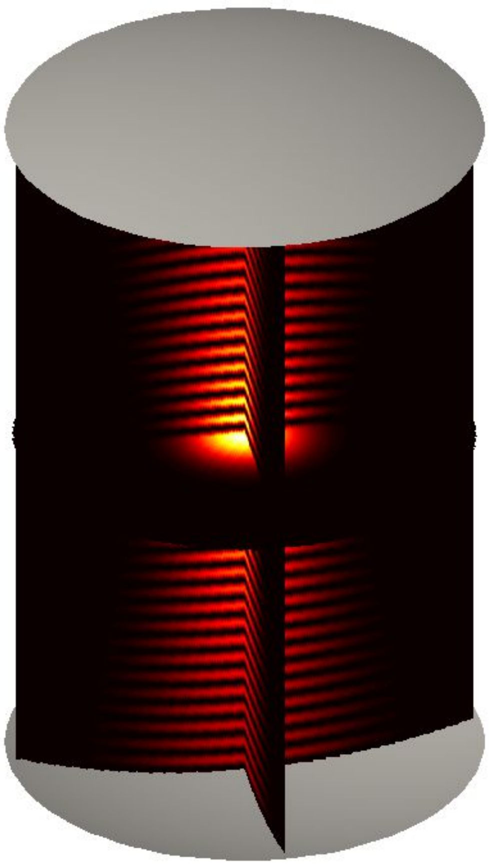
A TEM_{00} cavity mode



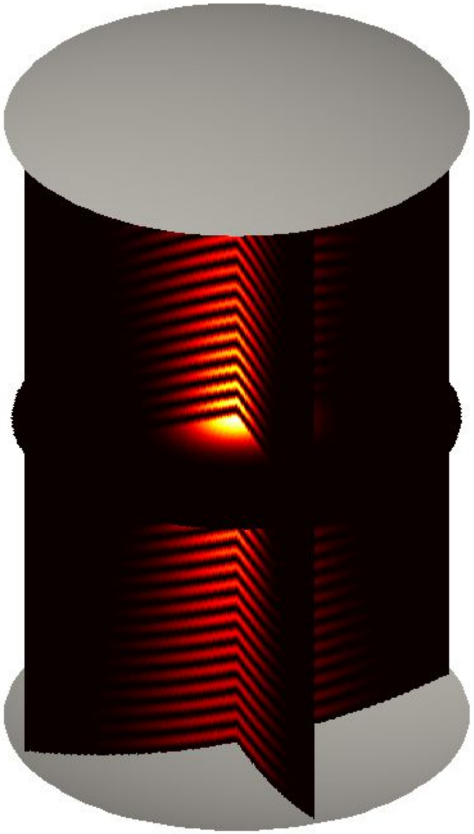
A TEM_{00} cavity mode



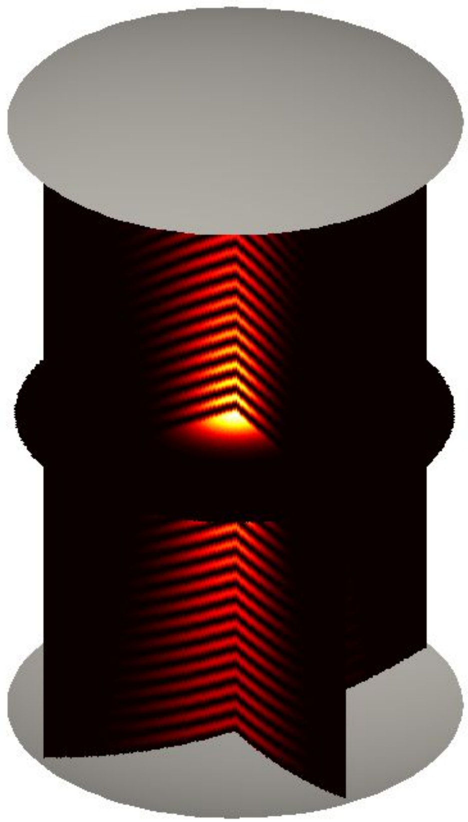
A TEM_{00} cavity mode



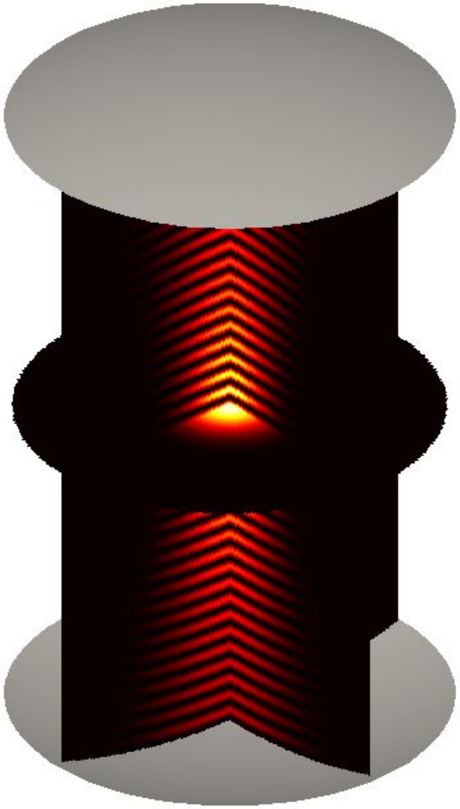
A TEM_{00} cavity mode



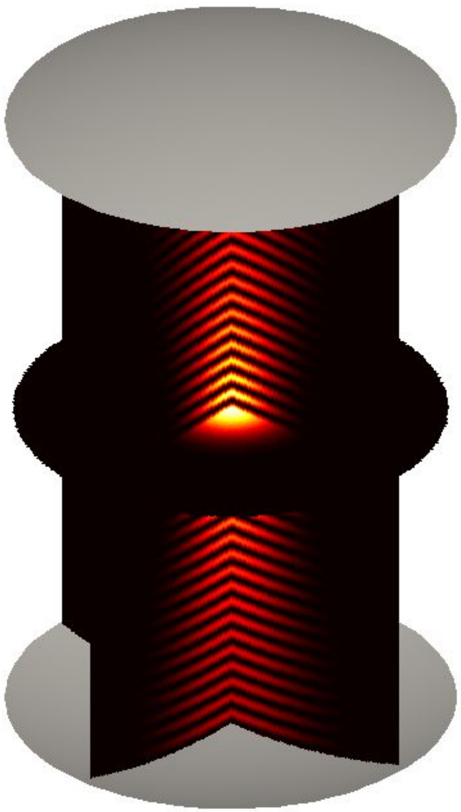
A TEM_{00} cavity mode



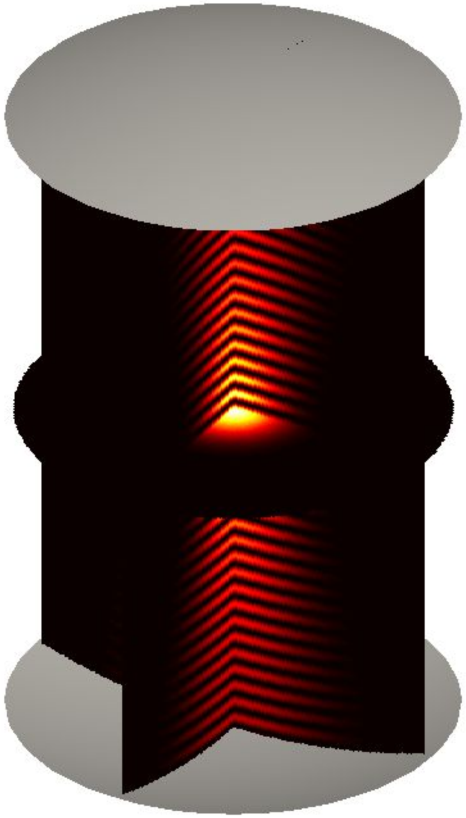
A TEM_{00} cavity mode



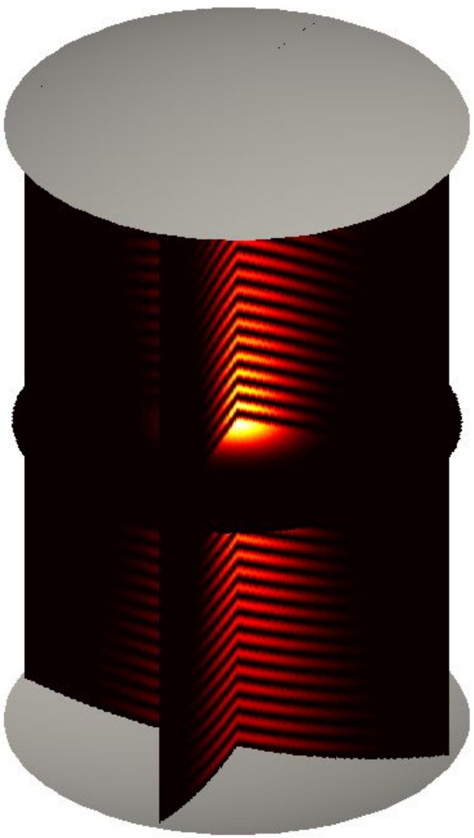
A TEM_{00} cavity mode



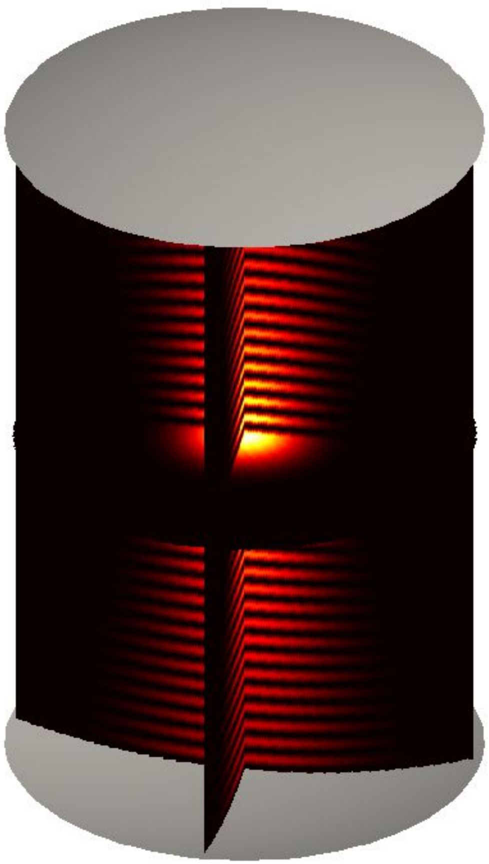
A TEM_{00} cavity mode



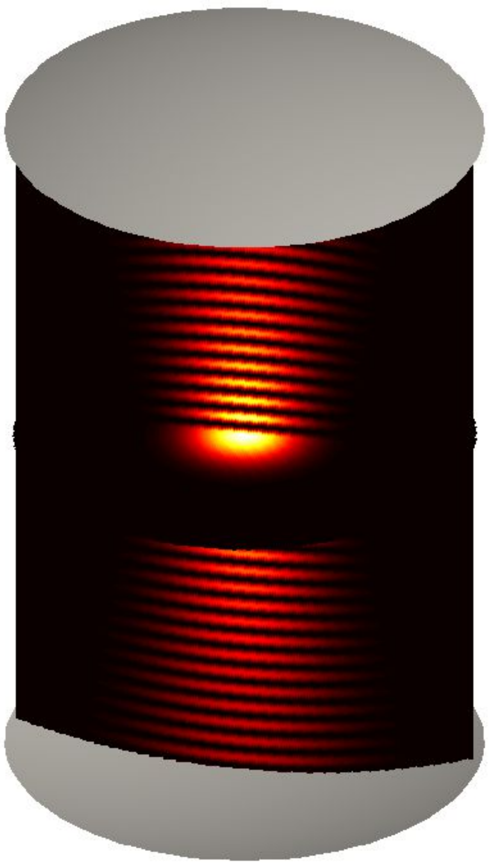
A TEM_{00} cavity mode



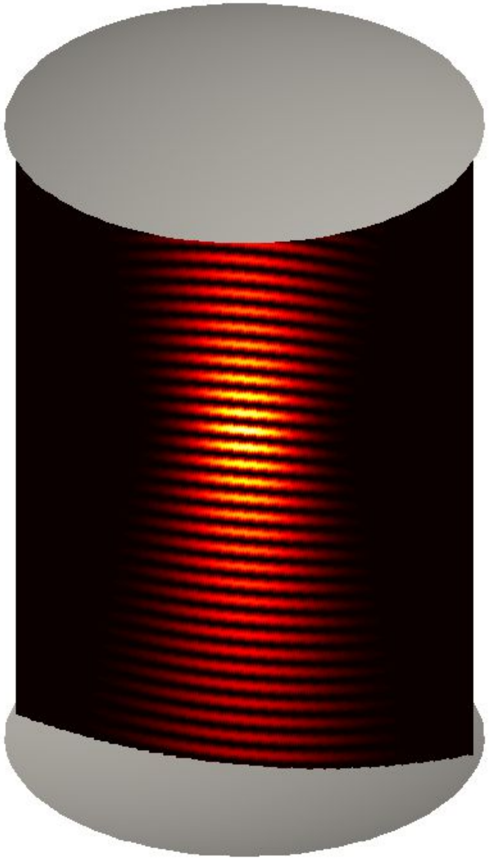
A TEM_{00} cavity mode



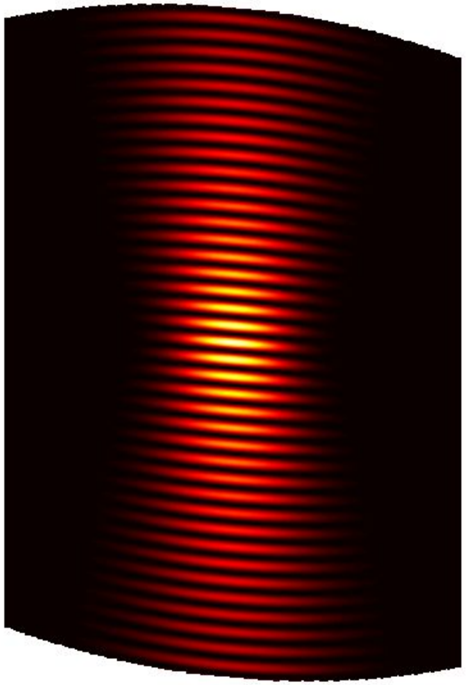
A TEM_{00} cavity mode



A TEM_{00} cavity mode

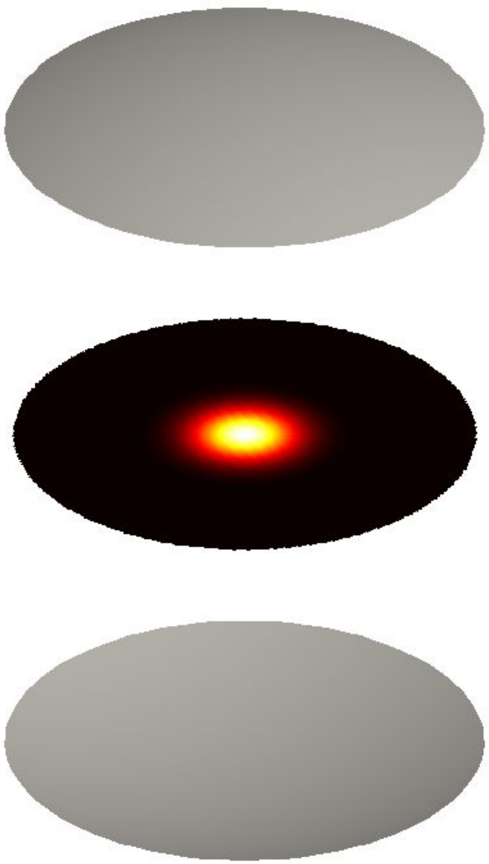


A TEM_{00} cavity mode



Field: $|E|^2$
R=20μm
λ_0=1μm
Confocal

A TEM_{00} cavity mode



Field: $|E|^2$
R=20μm
λ_0=1μm
Confocal

Mode frequencies

TEM$_{lm}$

Fresnel-Kirchoff integral for paraxial wave through a general ABCD system

$$u(x,y,z) = \frac{j}{B\lambda}\iint_S u(x_1,y_1,z_1)\exp\left[-jk\frac{A(x_1^2+y_1^2)+D(x^2+y^2)-2x_1x-2y_1y}{2B}\right]dx_1dy_1$$

→ TEM$_{lm}$ modes

That is, if $u(x_1,y_1,z_1) = H_l\left(\frac{\sqrt{2}x_1}{w_1}\right)H_m\left(\frac{\sqrt{2}y_1}{w_1}\right)\exp\left(-jk\frac{x_1^2+y_1^2}{2q_1}\right)$

then $u(x,y,z) = \left(\frac{1}{A+\frac{B}{q_1}}\right)^{1+l+m} H_l\left(\frac{\sqrt{2}x}{w}\right)H_m\left(\frac{\sqrt{2}y}{w}\right)\exp\left(-jk\frac{x^2+y^2}{2q}\right)$

After a round trip in cavity q=q$_1$. With round-trip ABCD matrix

$$A+\frac{B}{q} = A - jB\sqrt{-\frac{C}{B}} = \exp(-j\phi)$$

which has a unit amplitude and a phase $\phi = 2\cos^{-1}\pm\sqrt{A_1D_1}$

Total phase change for a round trip: $2kL-(1+l+m)\phi$

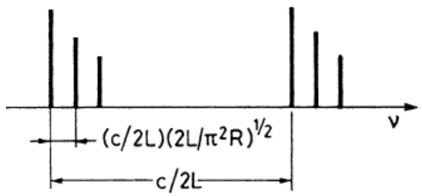
which must be $n\cdot 2\pi$: $\nu_{lmn} = \frac{c}{2L}\left(n+\frac{1+l+m}{\pi}\cos^{-1}\pm\sqrt{A_1D_1}\right)$

KTH VETENSKAP OCH KONST

Mode frequencies (special cases)

For two-mirror cavities: $$\nu_{lmn} = \frac{c}{2L}\left(n + \frac{1+l+m}{\pi}\cos^{-1} \pm\sqrt{g_1 g_2}\right)$$

- Near-planar resonator $$\nu_{lmn} = \frac{c}{2L}\left[n + \frac{(1+l+m)}{\pi}\left(\frac{2L}{R}\right)^{1/2}\right]$$

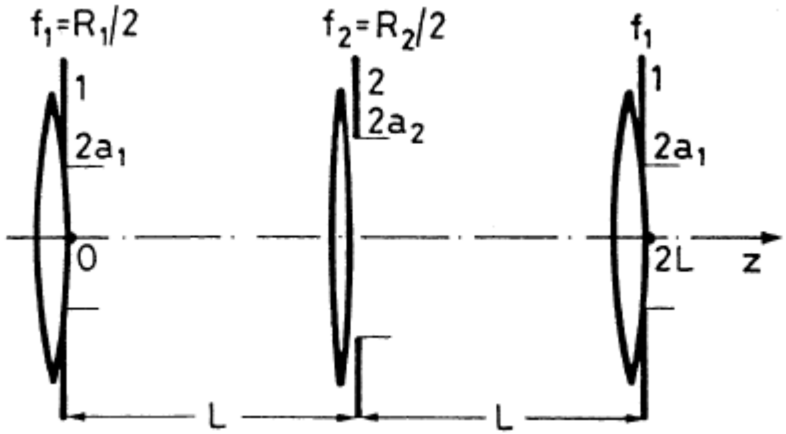


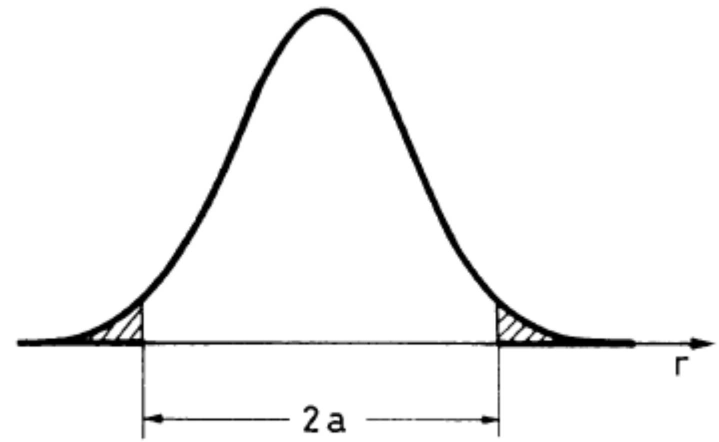
- Confocal resonator $$\nu_{lmn} = \frac{c}{4L}\left[2n + (1+l+m)\right]$$

- Concentric resonator $$\nu_{lmn} = \frac{c}{2L}\left(n + \frac{1+l+m}{\pi}\right)$$

Finite aperture

Purpose: modal discrimination





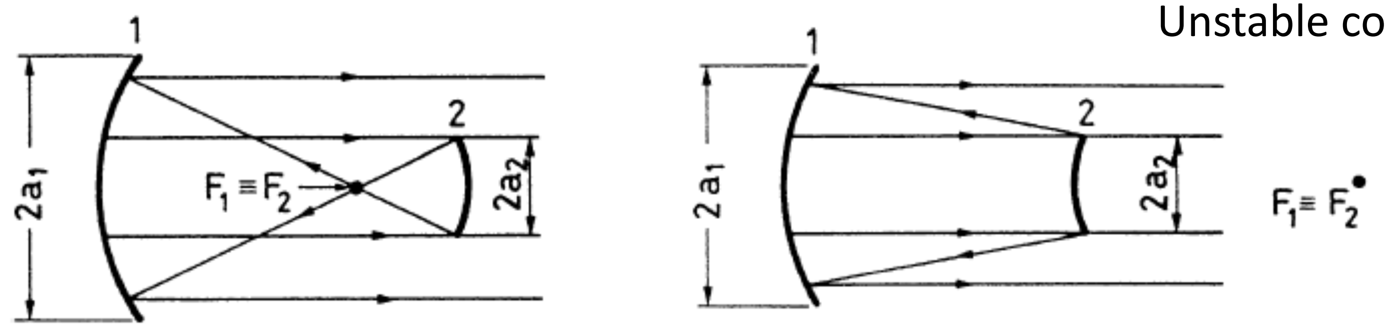
Single-transverse-mode condition (approximately):

Fresnel number $$N = \frac{a^2}{L\lambda} < 2$$

For con-focal case: $a < w_c$

Contents

Unstable resonators: Hard-edge



Unstable confocal

Purpose: large mode area while less modes

Round-trip loss $$L_i = \frac{M^2 - 1}{M^2}$$

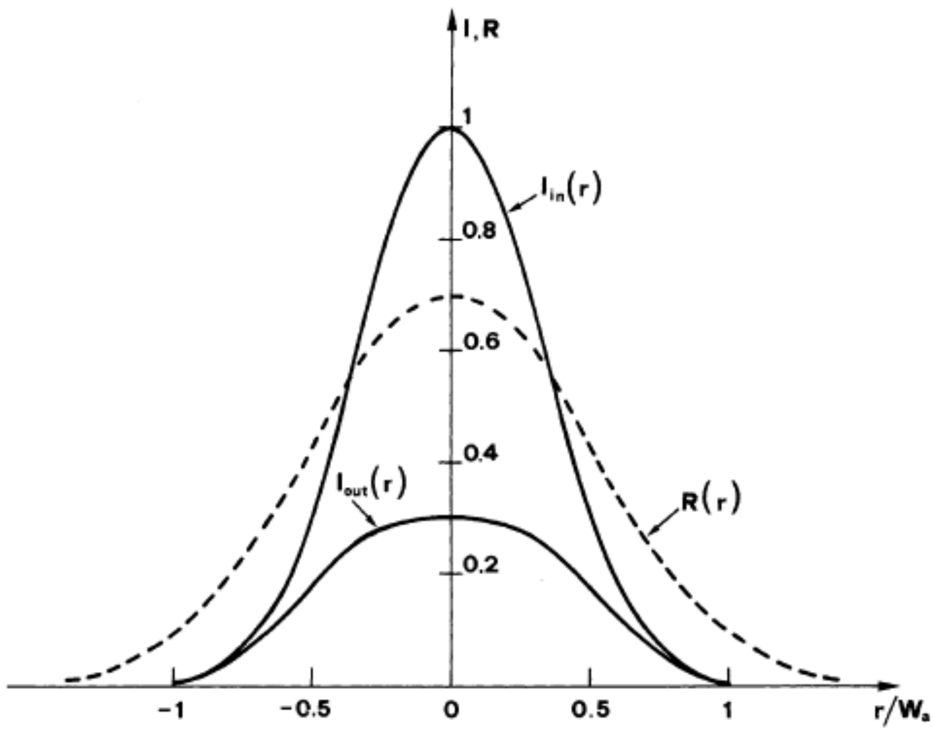
M: field spot magnification (dependent on g_1 and g_2)

Disadvantage:

- Annular-ring beam shape (doughnut)
- High-order annular rings in beam exist
- Sensitive to perturbation

Unstable resonators: Soft-edge

Mirrors with radially-varying reflectivity, with a profile in Gaussian or even super-Gaussian shape.

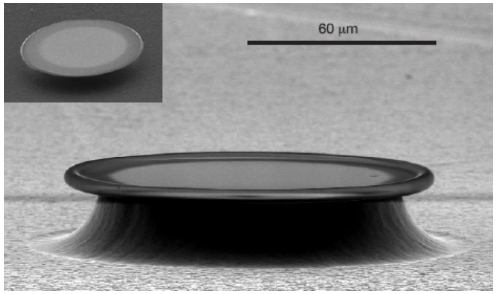


Contents

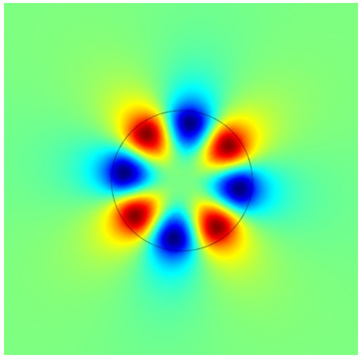
One more slide

- Fiber cavity

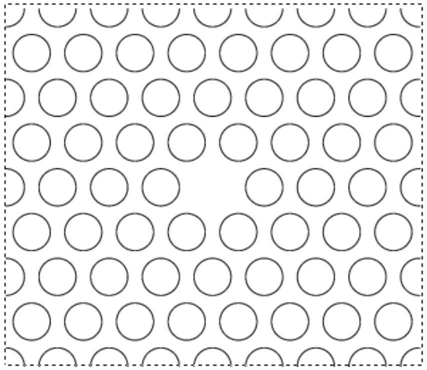
- Microdisk cavity



Q>10^8



- Photonic crystal cavity



Microdisk SEM: D.K. Armani et al., Nature 421, p.925 (2003).