Hybrid Conference

Quantum graphs
in
Mathematics, Physics and Applications

QGraph Network Meeting

Stockholm University, 6–8 December 2024

Programme

All ordinary lectures of the meeting will take place at Stockholm University (Cramér room), and on Zoom via

`https://stockholmuniversity.zoom.us/j/67006632913`

The schedule refers to Stockholm time (CET).

Friday, 6 December

12.30-13.30	**Lunch** (restaurant Provianten, Hus 2, floor 3)
14.00-14.25	R. Band
14.30-14.55	G. Sofer (ZOOM)
15.00-15.25	A. Brolin
15.30-16.00	**Coffee break**
16.00-16.25	A. Kostenko
16.30-16.55	M. Malamud (ZOOM)
17.00-17.25	L. Alon (ZOOM)
17.30-17.55	R. Carlson (ZOOM)
18.00-20.30	**Dinner** (at the department)

Saturday, 7 December

09.30-09.55	P. Exner
10.00-10.25	J. Lipovsky
10.30-11.00	**Coffee break**
11.00-11.25	M. Nowaczyk
11.30-11.55	M.E. Pistol
12.00-13.30	**Lunch** (at the department)
13.30-13.55	M. Kramar Fijavž
14.00-14.25	P. Bifulco
14.30-14.55	L. Baptista
15.00-15.30	D. Mugnolo
15.30-15.55	**Coffee break**
16.00-16.25	S. Özcan
16.30-16.55	V. Pivovarchik
17.00-17.25	D. Krejcirik
17.30-17.55	J. Kennedy
18.00-21.30	**Conference dinner (at the department)**

Sunday, 8 December

09.05-10.20	Nobel lectures in physics (Hopfield, Hinton) in Aula Magna (Stockholm Univ.) and via `https://www.kva.se/evenemang/the-nobel-lectures-2024/`
10.50-12.40	Nobel lectures in chemistry (Baker, Hassabis, Jumper)
14.00-15.50	Nobel lectures in economics (Acemoglu, Johnson, Robinson)

The Nodal Count Distribution: From Quantum Graphs to Discrete Graphs.

L. Alon (MIT Cambridge)

The nodal count of the k-th eigenfunction of a quantum graph represents the number of points where the eigenfunction vanishes. As k approaches infinity, this nodal count grows linearly with bounded fluctuations. In 2003, Smilansky et al. conjectured that these fluctuations, for large connected graphs with incommensurate edge lengths, follow a Gaussian distribution—a hypothesis inspired by the paradigm of quantum chaos. I will discuss previous work that has proven this conjecture for certain families of graphs. A similar question arises for discrete Schrödinger operators on graphs, where the nodal count is the number of edges on which an eigenfunction changes sign. This concept can be extended to any real symmetric matrix supported on a graph. I will present recent findings, both positive and negative results, regarding the universal Gaussian behavior of the nodal count fluctuations. This talk is based on joint work with Ram Band, Gregory Berkolaiko, Mark Goresky, and John Urschel.

Spectral statistics of quantum graphs with preferred orientation.

R. Band (Potsdam/Haifa)

Quantum graphs with vertex conditions of preferred orientation have been introduced by Exner and Tater a few years ago. Since then they are an interesting topic of study both theoretically and from the applied perspective. We are interested in studying the spectral statistics of such graphs from the viewpoint of the Bohigas-Giannoni-Schmit conjecture. Motivated by this conjecture, we compare the spectral statistics of preferred orientation graphs with the corresponding RMT ensembles, and along this way discover two surprises...

This is a joint work with Aviya Cohen, Pavel Exner and Divya Goel

Mean Distance on Metric Graphs.

L. Baptista (Lisbon)

We will introduce a concept of mean distance on metric graphs, which is a continuous version of the mean (or average) distance for discrete graphs. We start by presenting some examples and basic properties, such as some surgery principles, to illustrate its general behavior.

We will show that it behaves in a very similar way to the spectral gap (first non-zero eigenvalue) of the Laplacian with standard vertex conditions. We will also explore bounds connecting the mean distance and the spectral gap, which mirror bounds established for their counterparts on discrete graphs.

This is based on joint work with James Kennedy and Delio Mugnolo.

The heat content on compact quantum graphs and a combinatorial formula.

P. Bifulco (FernUniversität in Hagen)

We present the heat content $\mathcal{Q}_t(\mathcal{G})$ on metric graphs which is given by the L^1 norm of the heat semigroup applied to the constant $\mathbf{1}$-function, thus leading to a time-dependent quantity. We give a brief introduction into the heat content of metric graphs and list some basic properties which were already known for domains and manifolds, afterwards. Using then the so-called *path-sum-formula* for the heat kernel on metric graphs which is originally due to J.P. Roth, we derive a combinatorial expansion for the heat content leading to a *small-time asymptotic* for the heat content. If time permits, we also present an extremal Faber–Krahn inequality for small and large times which is related to classical results known for the *first eigenvalue* or the *torsional rigidity* of metric graphs.

This talk will be based on ongoing work together with Delio Mugnolo (Hagen).

Colin de Verdiere Parameter for Metric Graphs.

A. Brolin (Stockholm)

The classical Colin de Verdiere Parameter is a number assigned to discrete graphs. The number is of interest as it can tell us certain things about the topology of the graph, such as if the graph is planar or not. It is defined by looking at the maximal multiplicity of the second eigenvalue over a family of Laplacian matrices on the graph.

The Colin de Verdiere Parameter can also be obtained on metric graphs by looking at the maximal multiplicity of the second eigenvalue over a family of Laplacians with delta couplings at the vertices.

This is joint work with Pavel Kurasov.

A Quantum Graph FFT with applications to partial differential equations on networks.

R. Carlson (Colorado Springs)

Many natural and manufactured structures can be effectively modeled as networks of one dimensional segments joined at nodes. A new algorithm for the numerical solution of various time dependent partial differential equations on some of these networks is presented. The main novelty is a network version of the Fast Fourier Transform, which provides an efficient technique for expansions with eigenfunctions of the Laplace operator.

Lattice graphs violating time-reversal symmetry.

P. Exner (Doppler Institute, Prague)

We discuss spectral properties of infinite lattice-type quantum graphs focusing on effects coming the vertex couplings that violate the time-reversal invariance. In particular, we show that planar graphs with a circulant-type coupling may exhibit a nontrivial $\mathcal{PT}$-symmetry. As examples, we consider Kagome and Cairo lattices. We also find the spectrum of a square lattice in a homogeneous magnetic field with the aim to show how the two mechanisms violating the time-reversal invariance compete mutually.

Optimising the fundamental gap of a quantum graph.

J. Kennedy (Lisbon)

The gap conjecture, proved about 15 years ago by Andrews and Clutterbuck, asserts that the fundamental eigenvalue gap of a Schrödinger operator on a convex domain of fixed diameter with a convex potential, is minimised in the degenerate limit by the Schrödinger operator on an interval of the same diameter, with constant potential. This generalises a roughly 30-year-old result of Lavine, for Schrödinger operators with convex potentials on intervals.

Here we explore what kinds of results can hold in the setting of Schrödinger operators on compact metric trees (the most natural graph analogue of convex domains, where convex potentials can be defined in a natural way). We show that, in general, lower bounds analogous to those on domains are not possible.

More precisely, if one fixes the diameter of the graphs and an upper bound on the L^∞-norm of the potentials, then there is still a sequence of graphs whose fundamental gap converges to zero; while even on a fixed graph, one can find a sequence of convex potentials (whose L^∞-norm explodes) such that the fundamental gap converges to zero.

However, on a given graph, if one restricts to potentials whose L^∞-norm satisfies an a priori bound, then general compactness results based on Helly's theorem allow one to recover minimising and maximising convex potentials. In this case, the minimisers will be piecewise linear, but not constant in general. In fact, the constant potential being a minimiser seems to be a "rare" property in some sense, which we will try to make more precise in the talk.

This is based on joint work with Mohammed Ahrami, Zakaria El Allali, and Evans Harrell.

Laplacians on infinite graphs: discrete vs continuous.

A. Kostenko (Ljubljana/Vienna)

There are two different notions of a Laplacian operator associated with graphs: discrete graph Laplacians and continuous Laplacians on metric graphs (widely known as quantum graphs). One of our main messages is that these two settings should be regarded as complementary (rather than opposite) and exactly their interplay leads to important further insight on both sides. The main focus in this talk will be on intrinsic metrics on graphs.

Based on joint work with N. Nicolussi.

On a non-autonomous transport problem on a metric graph.

M. Kramar Fijavž (Ljubljana)

We shall consider a transport process on a metric graph with time-dependent velocities. We will use evolution families and evolution semigroups to show well-posedness of the corresponding non-autonomous abstract Cauchy problem. The talk is based on the joint work:

Budde, C., Kramar Fijavž, M. Well-posedness of non-autonomous transport equation on metric graphs. Semigroup Forum 108, 319–334 (2024)

The wave equation with Dirac damping on a non-compact star graph.

D. Krejcirik (Prague)

We consider the wave equation on non-compact star graphs, subject to a distributional damping defined through a Robin-type vertex condition with complex coupling. It is shown that the non-self-adjoint generator of the evolution problem admits an abrupt change in its spectral properties for a special coupling related to the number of graph edges. As an application, we show that the evolution problem is highly unstable for the critical couplings.

The talk is based on joint work with Julien Royer published in the Proceedings of the American Mathematical Society in 2023.

Magnetic quantum graphs with preferred-orientation coupling.

J. Lipovsky (Lund)

We introduce the coupling condition of the preferred orientation that was constructed by Exner and Tater to describe the anomalous quantum Hall effect. The properties of this condition heavily depend on the parity of the vertex – for vertices with odd degree, the transport through the vertex is difficult for large energies, while for even-degree vertices the particle can get through the vertex easily. I will mainly focus on the graphs in the magnetic field. In particular, I will discuss the spectral properties of the magnetic ring chain graphs and magnetic square lattice with preferred-orientation coupling. Moreover, I will mention the Band-Berkolaiko universality property for the ring chains and the Hofstadter's butterfly that appears for the square lattice.

Relations Between Spectral Properties of Metric and Discrete Graphs. Glazman–Povzner–Wienholtz theorem on graphs.

M. Malamud (RUDN)

We will discuss a close connection between spectral properties of a metric (quantum) graph with Kirchhoff or, more generally, δ-type couplings at vertices and the corresponding properties of a certain underlying weighted (discrete) graph.

In accordance with [1] certain spectral properties of both graphs either coincide or are closely related. Among them we mention the following ones:

- (1) self-adjointness;
- (2) lower semiboundedness;

- (3) discreteness property;
- (4) discreteness as well as finite dimensional properties of negative parts of their spectra;
- (5) certain spectral estimates (bounds for the bottom of spectra and essential spectra of graphs, CLR-type estimates etc.).

Recall that the Glazman–Povzner–Wienholtz theorem states that the semiboundedness of a Schrödinger operator, when combined with suitable local regularity assumptions on its potential and the completeness of the underlying manifold, guarantees its essential self-adjointness. We will discuss an extension of this result to Schrödinger operators on graphs. We first discuss the corresponding theorem for Schrödinger operators on metric graphs, allowing in particular distributional potentials which are locally H^{-1}. Besides, we exploit just mentioned connections between Schrödinger operators on metric graphs and weighted graphs to discuss a discrete version of the Glazman–Povzner–Wienholtz theorem.

The talk is based on results published in [1], [2].

References

[1] P. Exner, A. Kostenko, M. Malamud, H. Neidhardt, Spectral theory of infinite quantum graphs, Ann. Henri Poincare 19(11) (2018), 3457–3510.

[2] A. Kostenko, M. Malamud, N. Nicolussi, A Glazman–Povzner–Wienholtz theorem on graphs, Advances in Mathematics 395 (2022), 108158.

The torsional rigidity of graphs.

D. Mugnolo (Hagen)

Originally introduced by Pólya, the torsional rigidity of a domain can be defined in many different, equivalent ways, based on solutions of elliptic or parabolic equations; its main interest lies in the fact that it encodes relevant geometric properties of the domain, much like the ground state energy. In this talk I am going to introduce the torsional rigidity of a metric graph as well as the torsional rigidity of a combinatorial graph. These objects have a similar and, to some extent, parallel theory. After describing how to obtain geometric bounds on them using graph surgical tools, I will finally discuss the possibility of delivering bounds on the ground state energy and the Cheeger constant by means of the torsional rigidity.

This is based on joint work with Patrizio Bifulco and Marvin Plümer.

Spectrum of the Laplace operator on quantum graphs and defects in a graphene lattice.

M. Nowaczyk (Halmstad)

We consider the two-dimensional honeycomb structure of graphene that is a one-atom-thick layer of covalently bonded carbon atoms. The Laplace equation together with Kirchhoff boundary conditions at the nodes model the movement of low-energy free electrons in such a structure. Our research is based on the trace formula that combines the spectrum properties of the Laplace operator with the geometric properties of the underlying quantum graph. To be more specific, we use a one-to-one correspondence between the eigenvalues and the lengths of the closed paths. We investigate the four common types of defects in graphene, and based on the closed paths of odd lengths, we show the method for determining the type and the position of a defect.

This is a joint work with Margaret Archibald and Sonja Currie.

Isospectral quantum graphs.

S. Özcan (Dokuz Eylül University)

(joint with Matthias Täufer)

Initially, torsional rigidity is defined as the L^1-norm of the solution v of

$$\begin{cases} -\Delta v(x) = 1, & x \in \Omega \\ v(z) = 0, & z \in \partial\Omega \end{cases}$$

Pólya noticed that torsional rigidity is actually a geometric constant that depends on the shape and size of a given domain. In [2], Pólya demonstrated that among all open bounded domains with the same area, the circular domain possesses the greatest torsional rigidity.

In [1], Mugnolo and Plümer developed the theory of torsional rigidity for the Laplacian on metric graphs with at least one Dirichlet vertex. In this talk, we will mention the torsional rigidity for the Laplacian on metric graphs with δ-vertex conditions. We will establish upper and lower bounds for the torsional rigidity of graphs with the same total length.

References

[1] Mugnolo, D., Plümer, M., “On torsional rigidity and ground-state energy of compact quantum graphs.” Calc. Var. 62, (2023), 27.

[2] Pólya, G., “Torsional Rigidity, Principal Frequency, Electrostatic Capacity and Symmetrization.” Quarterly of Applied Mathematics, 6, (1948), 267–277.

Isospectral quantum graphs.

M.E. Pistol (Lund)

We have studied a variety of isospectral quantum graphs using computer algebra, where different boundary conditions have been imposed. We have restricted our studies to boundary conditions which are invariant under permutations of the edges. There are plenty of isospectral graphs under such boundary conditions. This is also the case when terminal vertices have Dirichlet and Neumann boundary conditions in a given graph. We will introduce the notion of “almost isospectrality”, where the zero eigenvalue is not considered. Finally we will show an example of two graphs which are isospectral duals, in a sense that will be made precise. Roughly, one can switch boundary conditions for the two graphs and they remain isospectral. Our software allows the visualisation of eigenfunctions for many types of boundary conditions and will be demonstrated.

Non-existence of co-spectral simple connected graphs with small number of edges.

V. Pivovarchik (Odesa/Vaasa)

We consider the Sturm-Liouville spectral problems on simple connected equilateral graphs with the standard conditions at the interior vertices and the Dirichlet conditions at the pendant vertices. We prove that if the number of edges does not exceed 7 then the asymptotics of eigenvalues of such problems uniquely determine the shape of the graph.

Coupled unidirectional chaotic microwave networks and quantum graphs.

L. Sirko (Warsaw)

A quantum graph Γ can be defined as a one-dimensional complex system with the Laplace operator $L(\Gamma) = -\frac{d^2}{dx^2}$ defined in the Hilbert space of square integrable functions. A quantum graph is constituted of one-dimensional leads

l_i which are connected at the vertices v_k. The propagation of a wave along a one-dimensional lead is described by the Schrödinger equation. The boundary conditions are applied to the wave functions as they enter and exit the vertices. Here, we will examine a frequently utilized standard (Neumann) boundary conditions. The Neumann boundary condition requires that the waves propagating in the leads meeting at the vertex v_k, are continuous and that the sum of outgoing derivatives at v_k is zero. The simulation of quantum graphs can be achieved through the use of microwave networks [1]. This is made possible by the equivalence of the stationary Schrödinger equation, which describes quantum graphs, and the telegraph equation, which describes microwave networks. In this report, we discuss the spectral statistics of nearly unidirectional quantum graphs and the transport properties of the coupled unidirectional microwave networks [2-3]. The resonances of the coupled unidirectional microwave networks are evaluated experimentally while the eigenvalues of the coupled unidirectional graphs are found numerically by using the pseudo-orbits method [4] which leads to the equation:

$$\det[I_{2N} - LS_G] = 0,$$

where I_{2N} is $2N \times 2N$ identity matrix, N is the number of the internal edges of the graph, L is the length matrix and S_G is the lead scattering matrix, containing scattering conditions at the graph vertices.

References

[1]. O. Hul, S. Bauch, P. Pakoński, N. Savytskyy, K. Życzkowski, and L. Sirko, "Experimental simulation of quantum graphs by microwave networks", Phys. Rev. E 69, 056209 (2004).

[2]. M. Akila and B. Gutkin "Spectral statistics of nearly unidirectional quantum graphs", J. Phys. A: Math. Theor. 48, 345101 (2015).

[3]. O. Farooq, A. Akhshani, M. Ławniczak, M. Białous, and L. Sirko, "Coupled unidirectional chaotic microwave graphs", Phys. Rev. E 110, 014206 (2024).

[4]. M. Ławniczak, J. Lipovsky, and L. Sirko, "Non-Weyl Microwave Graphs", Phys. Rev. Lett. 122, 140503 (2019).

Spectral properties of almost periodic quantum graphs.

G. Sofer (Haifa)

Given a Schrödinger operator, we are often interested in its integrated density of states, which roughly measures the number of states per unit volume in the system below a given energy. Special attention is often given to the values the integrated density of states attains at spectral gaps, known as gap labels, which are of physical importance. For instance, in the integer quantum Hall effect, the gap labels correspond to the quantized values of the Hall conductance. However, predicting these gap labels often requires the use of complicated machinery, such as K-theory, making the proofs quite challenging.

In this talk, we present a more accessible approach to developing gap labeling theorems, based on a method developed by Johnson and Moser. This is done by computing the average winding number of the Prüfer angle for the associated generalized eigenfunctions. Mainly, we present a gap labeling theorem for almost-periodic quantum graphs, providing a simple way to predict the possible gap labels of Sturmian metric graphs and other almost-periodic graphs inspired by one-dimensional tilings. This is done using a concept from dynamical systems known as the Schwartzman group.

Based on joint work in progress with Ram Band.