$\mathcal{N} = 2^*$ SYM Theory at Strong Coupling

With K. Zarembo and J. Gordon, arXiv: 1408.6040

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Outline

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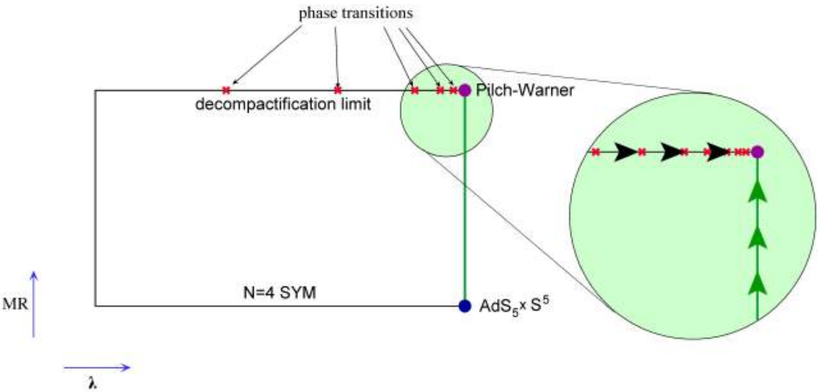
$\mathcal{N} = 2^*$ theory

- Unique massive deformation of $\mathcal{N} = 4$ SYM that preserves $\mathcal{N} = 2$ SUSY.
- Same field content as $\mathcal{N} = 4$ theory:
  - Hypermultiplet (**massive**):
  (2 complex scalar, 2 Majorana fermions)
  - Vector multiplet:
  (1 vector, 1 complex scalar, 2 Majorana fermions)
- Adjoint representation of $SU(N)$ gauge group.

Why study $\mathcal{N} = 2^*$ SYM?

- Supersymmetric localization on $\mathbb{S}^4$
  [Pestun, '12]
- Known holographic dual
  [Pilch Warner, '00] [Bobev et al., '13]
- Holographic principle tested at leading order for the circular Wilson loop
  [Buchel Russo Zarembo, '13]
- Interesting phase transitions on $\mathbb{R}^4$
  [Russo Zarembo, '13]

Phase diagram of $\mathcal{N} = 2^*$ on S^4



What do we want to understand?

- Does the flat space limit commute with the strong-coupling limit?
  Not trivial due to the phase structure.
- How can we probe the phase structures at the strong coupling?
- The result at leading order does **not** give the right prefactor for the Wilson loop!

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Localization technique

Property

Path integral is 1-loop exact!

Main idea

1. Deform the path integral such that it does **not** depend on the deformation parameter (α).
2. Expand the fields, but their fluctuations vanish when $\alpha \to \infty$.
3. Saddle-point method becomes exact!

$\mathcal{N} = 2^*$ on S^4

The partition function reduces to a finite dimensional integral of an **effective matrix model** [Pestun, '12]:

$$Z = \int d^{N-1}a \; \mathcal{Z}_{1-loop}(a) \; |\mathcal{Z}_{inst}(a)|^2 \; e^{-S_{classical}(a)}$$

as the scalar in the vector multiplet localizes at:

$$\langle \Phi \rangle = \mathrm{diag}(a_1, \dots, a_N)$$

breaking the original $SU(N)$ to $U(1)^{N-1}$.

Explicit expressions

- $S_{classical}(a) = \frac{8\pi^2 N}{\lambda} \sum_{j=1}^{N} a_j^2 \quad ; \quad \lambda = g_{YM}^2 N$
- $\mathcal{Z}_{1-loop}(a) = \prod_{i<j} \frac{(a_i - a_j)^2 H^2(a_i - a_j)}{H(a_i - a_j - M) H(a_i - a_j + M)}$
- $H(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}}$

Note: we set $R = 1$. To recover it, replace: $M \to MR$

At large N limit

- Instantons' contribution are exponentially suppressed:

$$|\mathcal{Z}_{inst}(a)|^2 \longrightarrow 1$$

- Use the saddle point approximation:

$$Z = \int d^{N-1}a \; e^{-N^2 S_{eff}(a)} \quad ; \quad \frac{\partial S_{eff}}{\partial a_i} = 0$$

Saddle point equation

$$\boxed{\frac{\partial S_{eff}}{\partial a_i} = 0 \quad \Rightarrow \quad \frac{1}{N}\sum_{i\neq j} S(a_i - a_j) = \frac{8\pi^2}{\lambda}}$$

- $S(x) \equiv \dfrac{1}{x} + \dfrac{1}{2} K(x+M) + \frac{1}{2} K(x-M) - K(x)$
- $K(x) \equiv -\dfrac{H'(x)}{H(x)} \quad ; \quad H(x) = \prod_{n=1}^{\infty} \left(1 + \dfrac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}}$

Continuous approximation

$$\boxed{\int_{-\mu}^{\mu} dy\, \rho(y) S(x-y) = \frac{8\pi^2}{\lambda} x}$$

- Large N master field, i.e. a density distribution:

$$\rho(x) = \frac{1}{N} \sum_{i}^{N} \delta(x - a_i)$$

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Large λ

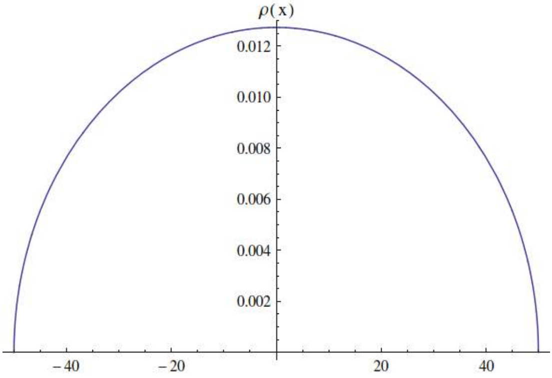
The saddle point equation for $\lambda \to \infty$:

$$\fint_{-\mu}^{\mu} dy\, \rho(y) \frac{1+M^2}{x-y} = \frac{8\pi^2}{\lambda} x$$

Solved by Wigner's semi-circular distribution:

$$\rho(x) = \frac{2}{\pi\mu^2}\sqrt{\mu^2 - x^2} \quad ; \quad \mu = \frac{\sqrt{\lambda(1+M^2)}}{2\pi}$$

Wigner's semi-circular distribution



Observable: Circular Wilson loop

$$W = \langle \frac{1}{N} P \exp\left(\oint ds \left(i\, \dot{x}^{\mu} A_{\mu} + |\dot{x}| \Phi \right) \right) \rangle$$

It is mapped to:

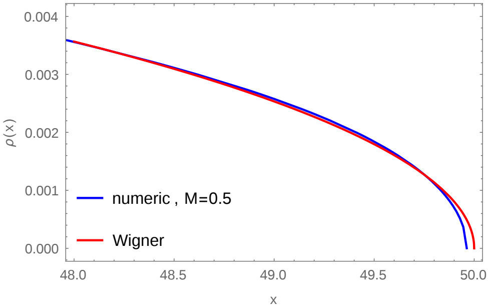
$$W = \langle \frac{1}{N} \sum_{i}^{N} e^{2\pi a_i} \rangle = \int_{-\mu}^{\mu} ds\, \rho(x)\, e^{2\pi x}$$

The leading order result recovers the **perimeter law**, in agreement with the dual theory [Buchel Russo Zarembo, '13]:

$$W \approx e^{2\pi\mu} \Rightarrow \boxed{\log W = \sqrt{\lambda(1 + (MR)^2)} + O(\lambda^0) \to \sqrt{\lambda}\, MR}$$

Wrong endpoint distribution

- The solution is not good close to the endpoint!
- For the Wilson loop, the prefactor to the exponential needs to be corrected!



Density close to the endpoints

Endpoint distribution, for $\xi \equiv \mu - x \sim 1$:

$$\rho(\xi) = \frac{2^{\frac{3}{2}}}{\pi \mu^{\frac{3}{2}}} f(\xi)$$

Need to match the asymptotics of the bulk distribution:

$$f(\xi) \xrightarrow{\xi \gg 1} \sqrt{\xi}$$

Saddle point equation

- Subtract the contribution from the bulk
- Regularize the function with $g(\xi) = f(\xi) - \sqrt{\xi}$

$$\boxed{\int_0^\infty d\eta\, g(\eta) S(\eta - \xi) = F(\xi)}$$

$$F(\xi) \equiv \int_0^\infty d\eta \left(\frac{1 + M^2}{\eta - \xi} - S(\eta - \xi) \right)$$

The Wiener-Hopf Method

1. Convolution: $(S * g)(\xi) = F(\xi) + \theta(-\xi)X(\xi)$
2. Fourier space: $\hat{S}(\omega)\hat{g}_+(\omega) = \hat{F}(\omega) + \hat{X}_-(\omega)$
3. Factorize the kernel: $\hat{S}(\omega) = \frac{1}{G_+(\omega)G_-(\omega)}$

$$\frac{\hat{g}_+(\omega)}{G_+(\omega)} = G_-\hat{F}(\omega) + G_-\hat{X}_-(\omega)$$

4. Project out − terms: $\boxed{\hat{g}_+(\omega) = G_+(\omega)\left(G_-F\right)_+(\omega)}$

$\pm$ mean analytic in the upper/lower half complex plane.

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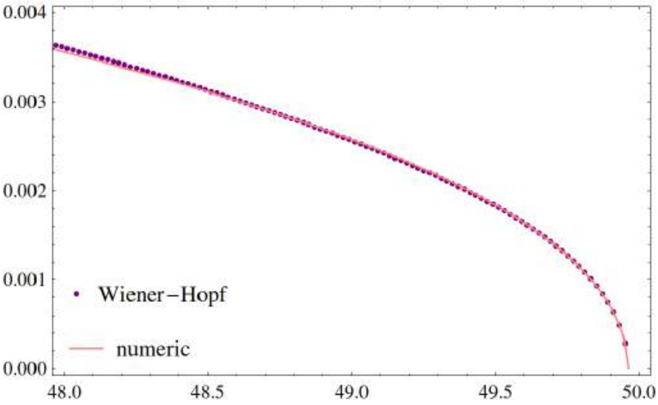
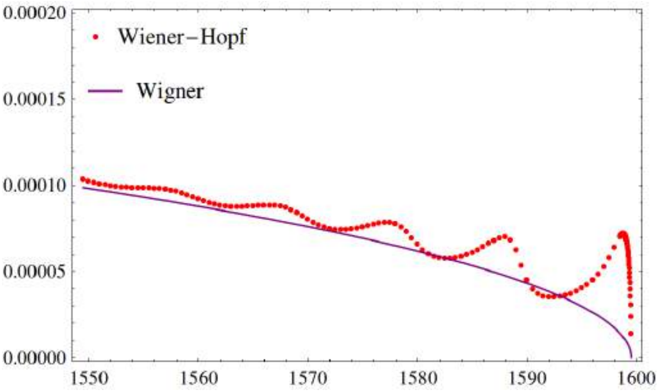
Wiener-Hopf solution

$$\begin{aligned}\hat{g}(\omega) \;=\; & \frac{i^{\frac{3}{2}}\sqrt{\pi}}{2\omega\sqrt{\omega+i\epsilon}}\left[\frac{M^2\sinh^2\frac{\omega}{2}-\sin^2\frac{M\omega}{2}}{\sinh^2\frac{\omega}{2}+\sin^2\frac{M\omega}{2}}\right.\\ & +\left(M^2+1\right)^2\omega\,\mathrm{e}^{-\frac{i\phi\omega}{2\pi}}\,\frac{\Gamma\left(\frac{M-i}{2\pi}\omega\right)\Gamma\left(-\frac{M+i}{2\pi}\omega\right)}{\Gamma^2\left(-\frac{i\omega}{2\pi}\right)}\\ & \left.\times\sum_{n=1}^{\infty}\frac{(-1)^n}{n n!}\left(\frac{\mathrm{e}^{\frac{i\phi n}{M-i}}}{\omega-\frac{2\pi n}{M-i}}\,\frac{\Gamma\left(\frac{M+i}{M-i}n\right)}{\Gamma^2\left(\frac{i}{M-i}n\right)}+\frac{\mathrm{e}^{-\frac{i\phi n}{M+i}}}{\omega+\frac{2\pi n}{M+i}}\,\frac{\Gamma\left(\frac{M-i}{M+i}n\right)}{\Gamma^2\left(-\frac{i}{M+i}n\right)}\right)\right]\end{aligned}$$

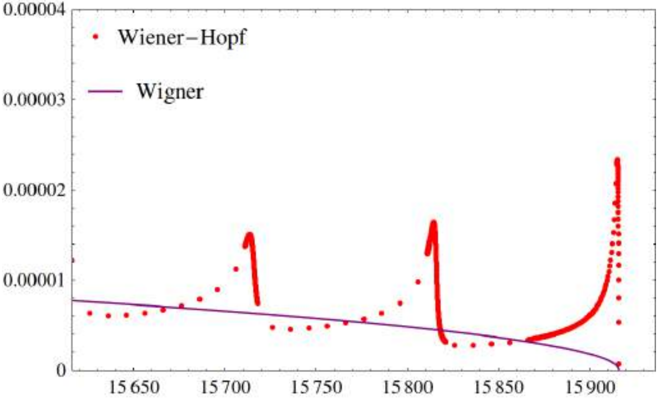
Density at endpoint, $M = 0.5$



Density at endpoint, $M = 10$



Density at endpoint, $M = 100$



Large M /decompactification limit

1) At the extreme endpoint:

$$\rho(\xi) = \frac{2^{3/2}}{\pi\mu^{3/2}} M\sqrt{\xi}$$

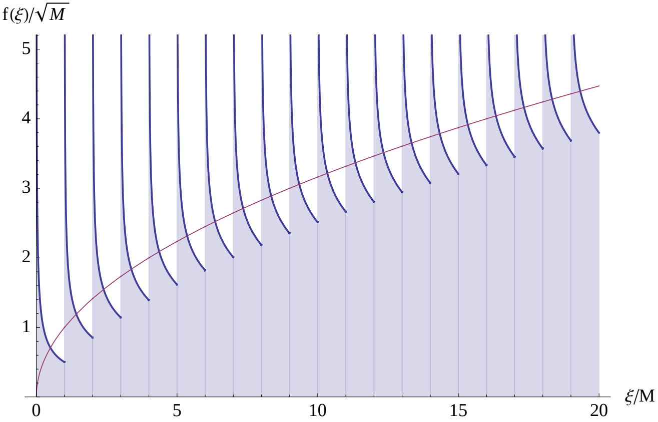
2) Oscillatory regime when $\xi \sim M$:

$$\rho(\xi) = \frac{1}{\pi}\sqrt{\frac{2M}{\mu^3}} \sum_{k=0}^{\left[\frac{\xi}{M}\right]} \frac{1}{\sqrt{\left\{\frac{\xi}{M}\right\} + k}}$$

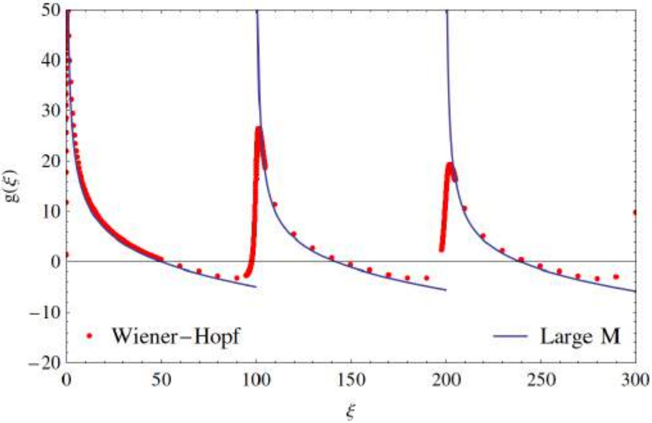
3) Matching regime with the semicircle, when $\xi \sim M^2$:

$$\rho(\xi) = \frac{2^{3/2}}{\pi\mu^{3/2}} \left(\sqrt{\xi} + \frac{M}{4\sqrt{\xi}}\right)$$

Oscillatory regime, $M \to \infty$



Comparing with $M = 100$



Other solutions

At large M:

- Wilson loop:

$$W = \sqrt{\frac{8\pi}{MR}}\,\lambda^{-\frac{3}{4}}\,\mathrm{e}^{\left(\sqrt{\lambda}-\pi\right)MR-2}$$

- Correction to the endpoint:

$$\mu = \frac{\sqrt{\lambda(1+M^2)}}{2\pi} - \frac{M}{2} + O(\lambda^{-1/2})$$

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Summary

- **Endpoint density distribution** at strong coupling for general *MR*
- At the decompactification limit, we saw the **phase transitions**!
- The decompactification and the strong coupling **limits commute**! *
- Correct prefactor for the **Wilson loop**
- Correction to the **endpoint**

*see also [Zarembo, '14]

What to do next?

- Test massive holography at the quantum level: compute **quantum corrections** for the Wilson loop, in the dual string theory.
- **Probe the phase structure** using other observables: high representation Wilson loops.
- Understand the phase transitions in the **dual theory**.

Thank you for your attention!