Computation of Robust Option Prices via Structured Martingale Optimal Transport

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based on joint work with ***Sigrid Källblad*** *and* ***Johan Karlsson***

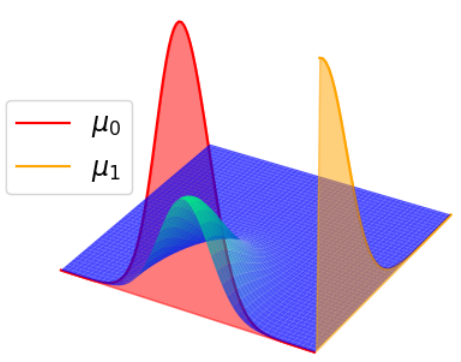
Overview

- Introduction and background
- Our method
- Numerical examples

Introduction and background

Introduction to structured optimal transport (OT)

- Let μ_t be given prob. measures on $\mathrm{X} \subseteq \mathbb{R}$ for $t = 0, 1, \dots, T$
- Suppose that X is contained in a finite number of points, n



$$\begin{aligned} \min_{Q \in \mathbb{R}_+^{n^{T+1}}} \quad & \langle C, Q \rangle \\ \text{subject to} \quad & P_t(Q) = \mu_t, \quad t = 0, 1, \dots, T \end{aligned} \tag{1}$$

Problem (1) is very large, n^{T+1} variables

Introduction to structured optimal transport (OT) II

- Entropic regularization to solve **bi-marginal** $(T = 1)$ problems for large n (Cuturi 2013)

$$\begin{aligned} \min_{Q \in \mathbb{R}_+^{n\times n}} \quad & \langle C, Q \rangle + \varepsilon D(Q) \\ \text{subject to} \quad & P_t(Q) = \mu_t, \quad t = 0, 1 \end{aligned}$$

$\Rightarrow$

$$u_t^{(k+1)} \leftarrow (\mu_t \odot u_t^{(k)}) \oslash P_t(Q^{(k)}), \; k = 1, 2, \dots$$

Introduction to structured optimal transport (OT) II

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$$\min_{Q\in\mathbb{R}_+^{n\times n}} \quad \langle C, Q\rangle + \varepsilon D(Q)$$
$$\text{subject to} \quad P_t(Q) = \mu_t, \quad t = 0, 1$$

$$\Rightarrow \quad u_t^{(k+1)} \leftarrow (\mu_t \odot u_t^{(k)}) \oslash P_t(Q^{(k)}), \; k = 1, 2, \ldots$$

- Not enough for **multi-marginal** $(T \geq 2)$ problems — must exploit sparse structures!

When C is of the form

$$C(i_0, \ldots, i_T) = \sum_{t=1}^{T} C_t(i_{t-1}, i_t), \quad i_0, \ldots, i_T = 1, \ldots, n$$

for $C_t \in \mathbb{R}^{n\times n}$ the projection $P_t(Q^{(k)}) = v_t \odot u_t^{(k)} \odot w_t$ for some vectors $v_t, w_t \in \mathbb{R}^n$
(Elvander, Haasler, Jakobsson, Karlsson 2020)

The martingale optimal transport (MOT) problem

Our setting:

- Let $(\Omega, \mathscr{F}, \mathbb{Q}, S)$ refer to a a probability space $(\Omega, \mathscr{F}, \mathbb{Q})$ together with a stochastic process $S : \Omega \times \{0, 1, \dots, T\} \to \mathbb{R}$
- We consider $(\Omega, \mathscr{F}, \mathbb{Q}, S)$ a *market model* if S a $\mathbb{Q}$-martingale
  - suppose that the interest rate is zero
- Let μ_t for $t \in \mathscr{T} \subset \{0, 1, \dots, T\}$ be given marginals of S, i.e. $\mu_t = \text{Law}(S_t)$
  - suppose that the given marginals are in convex order

The martingale optimal transport (MOT) problem II

The MOT problem is an OT problem with an additional martingale constraint

$$
\begin{aligned}
\inf_{(\Omega,\mathscr{F},\mathbb{Q},S)} \quad & \mathbb{E}_{\mathbb{Q}}[\phi(S_0,\dots,S_T)] & \\
\text{subject to} \quad & S_t \sim_{\mathbb{Q}} \mu_t, & t \in \mathscr{T} \\
& \mathbb{E}_{\mathbb{Q}}[S_t \,|\, \sigma(S_0,\dots,S_{t-1})] = S_{t-1}, & t = 1,2,\dots,T
\end{aligned}
$$

- Introduced to address robust pricing ~ 10 years ago
  (Beiglböck, Henry-Labordère, Penkner 2013 & Galichon, Henry-Labordère, Touzi 2014)
- Entropic regularization on **bi-marginal** MOT problems (De March 2018)
- Note: martingale constraint links t^{th} marginal to all earlier marginals

Our method

Problem formulation

We consider problems with payoffs of the form

$$\phi(S_0, \dots, S_T) = \sum_{t=1}^{T} \phi_t(S_{t-1}, X_{t-1}, S_t, X_t), \quad X_t = \begin{cases} h_0(S_0), & t = 0 \\ h_t(S_{t-1}, X_{t-1}, S_t, X_t), & t = 1, \dots, T \end{cases}$$

Many payoff functions of financial interest belongs to this class. *Some examples are:*

Choice of X		**Example of derivative**
Rolling max	$X_t = \max_{0 \le j \le t}\{S_j\}$	Lookback options
Rolling mean	$X_t = (t+1)^{-1} \sum_{j=0}^{t} S_j$	Asian options
Realised variance	$X_t = t^{-1} \sum_{j=1}^{t} (\log(S_{j+1}/S_j))^2$	Variance swaps
Indicator	$X_t = \chi_{A_0 \times \dots \times A_t}(S_0, \dots, S_t)$	Barrier options
Sum of truncated rel. return	$X_t = \sum_{j=1}^{t} \max\{\min\{(S_j - S_{j-1})/S_j, C\}, 0\}$	Cliquet options
Counter	$X_t = \sum_{j=0}^{t} \chi_A(S_j)$	Parisian options

Problem reformulation — reduce the path dependency

$$\begin{aligned} \inf_{(\Omega,\mathscr{F},\mathbb{Q},S)} \quad & \sum_{t=1}^{T} \mathbb{E}_{\mathbb{Q}}[\phi_t(S_{t-1}, X_{t-1}, S_t, X_t)] & \\ \text{s.t.} \quad & S_t \sim_{\mathbb{Q}} \mu_t, & t \in \mathscr{T} \\ & \mathbb{E}_{\mathbb{Q}}[S_t \,|\, \sigma(S_0, \ldots, S_{t-1})] = S_{t-1}, & t = 1, \ldots, T \end{aligned} \tag{2}$$

$$\begin{aligned} \inf_{(\Omega,\mathscr{F},\mathbb{Q},S)} \quad & \sum_{t=1}^{T} \mathbb{E}_{\mathbb{Q}}[\phi_t(S_{t-1}, X_{t-1}, S_t, X_t)] & \\ \text{s.t.} \quad & S_t \sim_{\mathbb{Q}} \mu_t, & t \in \mathscr{T} \\ & \mathbb{E}_{\mathbb{Q}}[S_t \,|\, \sigma(S_{t-1}, X_{t-1})] = S_{t-1}, & t = 1, \ldots, T \end{aligned} \tag{3}$$

Theorem

The MOT problem (2) and the OT problem (3) are equivalent in the sense that any optimal solution of problem (2) is also an optimal solution to problem (3), while any optimal solution of problem (3) can be used to construct an optimal solution to problem (2). The problem values coincide.

Proof:

- Construct a Markov process $(\tilde{S}, \tilde{X})$ with the same marginal distributions as (S, X)
- Then compare the feasible sets

Formulate as an LP

$$
\begin{aligned}
\inf_{(\Omega,\mathscr{F},\mathbb{Q},S)} \quad & \sum_{t=1}^{T} \mathbb{E}_{\mathbb{Q}}[\phi_t(S_{t-1}, X_{t-1}, S_t, X_t)] & \\
\text{s.t.} \quad & S_t \sim_{\mathbb{Q}} \mu_t, & t \in \mathscr{T} \\
& \mathbb{E}_{\mathbb{Q}}[S_t \,|\, \sigma(S_{t-1}, X_{t-1})] = S_{t-1}, & t = 1, \dots, T
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\min_{Q \in \mathbb{R}_+^{(n_S n_X)^{T+1}}} \quad & \langle C, Q \rangle & \\
\text{s.t.} \quad & P_t^S(Q) = \mu_t, & t \in \mathscr{T} \\
& (P_{t-1,t}(Q) \odot \Delta)\mathbf{1}_{n_S n_X} = \mathbf{0}_{n_S n_X}, & t = 1, \dots, T
\end{aligned} \tag{4}
$$

where $C(i_0, \dots, i_T) = \sum_{t=1}^{T} \Phi_t(i_{t-1}, i_t) + I_t(i_{t-1}, i_t)$ for penalties I_t and $\Delta(i_{t-1}, i_t) = (\mathbf{1}_{n_X} \otimes s)(i_t) - (\mathbf{1}_{n_X} \otimes s)(i_{t-1})$

Proposition

Suppose that we restrict problem (3) to models such that the support of the price process at each time point is contained within $n_s \in \mathbb{N}$ points. Then problems (3) and (4) are equivalent.

Solving the LP — regularization and coordinate dual ascent

- Entropic regularization

$$\begin{aligned} \min_{Q\in\mathbb{R}_+^{(n_S n_X)^{T+1}}} \quad & \langle C, Q\rangle + \varepsilon D(Q) \\ \text{s.t.} \quad & P_t^S(Q) = \mu_t, && t \in \mathcal{T} \\ & (P_{t-1,t}(Q) \odot \Delta)\mathbf{1}_{n_S n_X} = \mathbf{0}_{n_S n_X}, && t = 1, \dots, T \end{aligned} \tag{5}$$

$D(Q) = \langle Q, \log(Q) - \mathbf{1}\rangle$
regularizing entropy term,
scaled with $\varepsilon > 0$ small

- The dual of the regularized problem

$$\max_{\lambda,\gamma} \sum_{t\in\mathcal{T}} \lambda_t^\top \mu_t - \varepsilon\langle K, U_\lambda \odot G_\gamma\rangle \tag{6}$$

$$K(i_0, \dots, i_T) = \prod_{t=1}^{T} K_t(i_{t-1}, i_t)$$

$$G_\gamma(i_0, \dots, i_T) = \prod_{t=1}^{T} G_t(i_{t-1}, i_t)$$

$$U_\lambda(i_0, \dots, i_T) = \prod_{t\in\mathcal{T}} (\mathbf{1}_{n_X} \otimes u_t)(i_t)$$

Solving the LP — regularization and coordinate dual ascent II

- Optimality conditions for the dual problem (6):

$$u_t = \mu_t \oslash P_t^S(K \odot U_\lambda^{-t} \odot G_\gamma), \qquad t \in \mathcal{T}$$
$$(P_{t,t+1}(K \odot U_\lambda \odot G_\gamma) \odot \Delta)\mathbf{1}_{n_S n_X} = \mathbf{0}_{n_S n_X}, \quad t = 0, 1, \dots, T-1$$

- The minimizing primal variable:

$$Q_{\lambda,\gamma} = K \odot U_\lambda \odot G_\gamma$$

Algorithm 1 High-level method

Initialise: $u_t \leftarrow \mathbf{1}_{n_S}$ for $t \in [T]$
$\gamma_t \leftarrow \mathbf{1}_{n_S n_X}$ for $t \in [T-1]$
while not converged **do**
  for $t \in T$ **do**
    $u_t \leftarrow \mu_t \oslash P_t^S(K \odot U_\lambda^{-t} \odot G_\gamma)$
  end for
  for $t = T-1, \dots, 0$ **do**
    find $\hat{\gamma}_t$ such that $(P_{t,t+1}(K \odot U_\lambda \odot G_\gamma) \odot \Delta)\mathbf{1}_{n_S n_X} = \mathbf{0}_{n_S n_X}$
    $\gamma_t \leftarrow \hat{\gamma}_t$
  end for
end while
$Q_{\lambda,\gamma} \leftarrow K \odot U_\lambda \odot G_\gamma$
return $Q_{\lambda,\gamma}$

Exploiting the structure for fast computation

Theorem

Define two families of help vectors $\hat{\psi}$ and ψ via the recursions

$$\hat{\psi}_t = \begin{cases} \mathbf{1}_{n_S n_X}, & t = 0 \\ (K_t \odot G_t)^\top (\hat{\psi}_{t-1} \odot \bar{u}_{t-1}), & t = 1, \dots, T \end{cases}, \quad \psi_t = \begin{cases} \mathbf{1}_{n_S n_X}, & t = T \\ (K_{t+1} \odot G_{t+1}) (\psi_{t+1} \odot \bar{u}_{t+1}), & t = 0, \dots, T-1 \end{cases}$$

where

$$\bar{u}_t = \begin{cases} \mathbf{1}_{n_X} \otimes u_t, & t \in \mathcal{T} \\ \mathbf{1}_{n_S n_X}, & t \in \{0, \dots, T\} \setminus \mathcal{T}. \end{cases}$$

Then λ and γ are optimal variables for the dual problem (6) if and only if the following equations hold

$$u_t = \mu_t \oslash P^S(\hat{\psi}_t \odot \psi_t), \quad t \in \mathcal{T},$$
$$\hat{\psi}_t \odot \bar{u}_t \odot (K_{t+1} \odot G_{t+1} \odot \Delta)(\psi_{t+1} \odot \bar{u}_{t+1}) = \mathbf{0}_{n_S n_X}, \quad t = 0, \dots, T-1.$$

Exploiting the structure for fast computation II

- Projections are replaced by matrix-vector products
- The help vectors ψ and $\hat{\psi}$ are defined recursively
- No need to form and store the full $(T+1)$-dimensional tensors K, U_λ, G_γ and $Q_{\lambda,\gamma}$

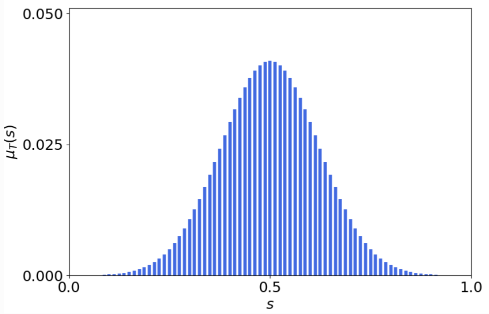
Once $Q_{\lambda,\gamma}$ optimal has been found, the projections $P_t(Q_{\lambda,\gamma})$ and $P_{t_1,t_2}(Q_{\lambda,\gamma})$ are recovered via matrix-vector products. Robust price obtained as

$$\langle \Phi, Q_{\lambda,\gamma} \rangle = \sum_{t=1}^{T} \langle \Phi_t, P_{t-1,t}(Q_{\lambda,\gamma}) \rangle.$$

Numerical examples

The maximum of the maximum

- Consider the maximum process, $X_t := \max_{j\in\{0,\dots,t\}} S_j$, $t = 0, 1, \dots, T$
- Suppose that μ_T and μ_0 are given:
  - μ_T is as given in the figure — it is centered in $\frac{1}{2}$
  - $\mu_0 = \delta_{\frac{1}{2}}$



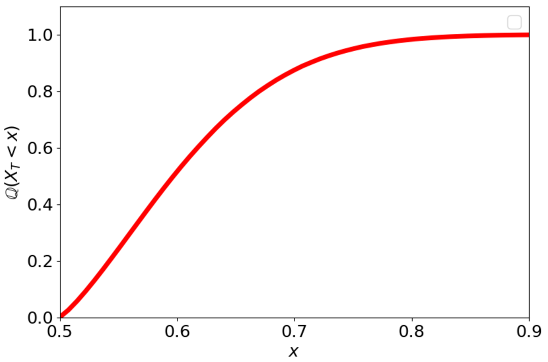
What's the law of X_T for the martingale model $(\Omega, \mathscr{F}, \mathbb{Q}, S)$ that maximizes $\mathbb{E}_{\mathbb{Q}}[X_T]$ while respecting μ_0 and μ_T?

The maximum of the maximum II

The corresponding continuous-time solution exists and is known (Hobson 1998)

- The law of the maximum for the maximizing continuous-time martingale model $(\Omega^*, \mathscr{F}^*, \mathbb{Q}^*, S^*)$ is (Brown, Hobson, Rogers 2001)

$$\mathbb{Q}^*(X_T^* \geq B) = \min_{0 \leq y \leq B} \frac{1}{B-y} \int (s-y)^+ \mathrm{d}\mu_T(s), \quad B > \frac{1}{2} \tag{7}$$



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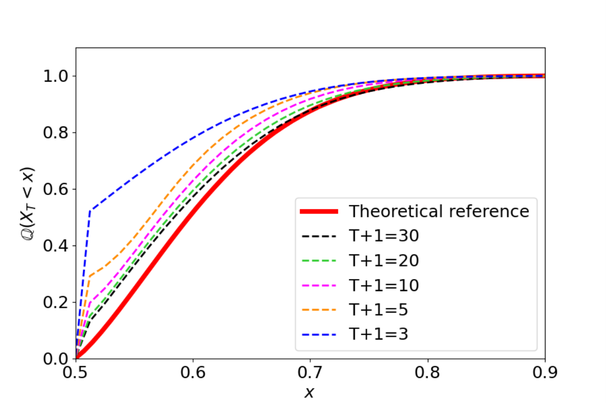

The robust price of a digital option

- Let μ_0 and μ_T be as in the previous example
- The payoff of a *digital option* with barrier $B > \frac{1}{2}$ is $\phi(S_0, \dots, S_T) = \chi_{[B,\infty)}(\max_{t\in\{0,\dots,T\}} S_t)$

What's the robust price of a digital option, considering **discrete-time** martingale models $(\Omega, \mathscr{F}, \mathbb{Q}, S)$ that respects μ_0 and μ_T?

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What's the robust price of a digital option, considering **discrete-time** martingale models $(\Omega, \mathscr{F}, \mathbb{Q}, S)$ that respects μ_0 and μ_T?

- For B fixed, enough to use $T = 2$ to obtain an equally optimal solution (Föllmer, Schied pp.416–419)
- Note that $\mathbb{E}_{\mathbb{Q}}[\chi_{[B,\infty)}(\max_{t\in\{0,\dots,T\}} S_t)] = \mathbb{Q}(\max_{t\in\{0,\dots,T\}} S_t \geq B)$ — cf. equation (7)

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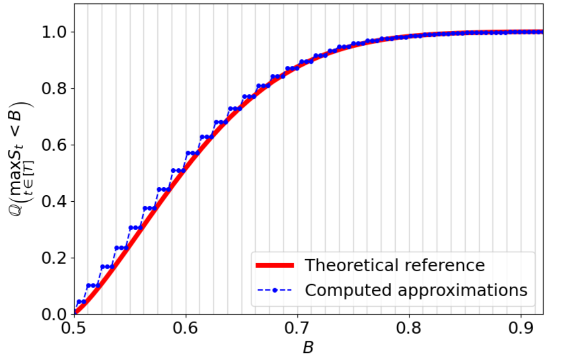
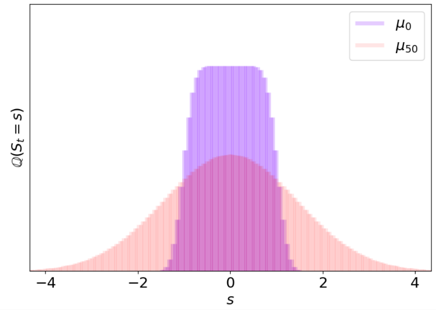
- By repeatedly optimizing for each individual B, we recover the law of the maximum from the previous example



Late and early transports

- For $T = 50$, let μ_0 and μ_{50} be given as in the figure
- Consider $\phi(S_0, \dots, S_T) = (T+1)^{-1} \sum_{t=0}^{T} S_t^2$ (arithmetic mean of a convex function)



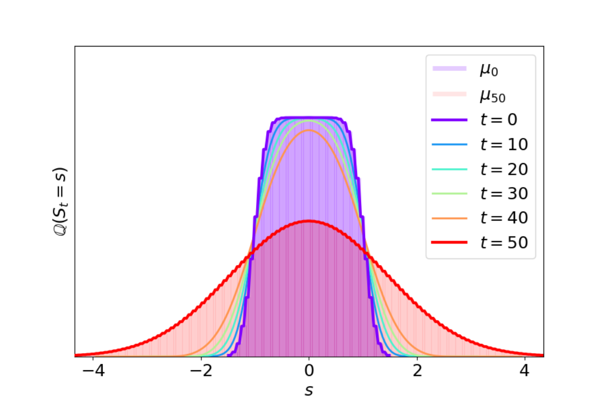
Since $\mathbb{E}_{\mathbb{Q}}[S_T^2] \geq \mathbb{E}_{\mathbb{Q}}[S_{T-1}^2] \geq \dots \geq \mathbb{E}_{\mathbb{Q}}[S_1^2] \geq \mathbb{E}_{\mathbb{Q}}[S_0^2]$,

$$\mathbb{E}_{\mathbb{Q}}[\phi(\underbrace{S_0, S_0, \dots, S_0, S_T}_{\text{“late transport”}})] \leq \mathbb{E}_{\mathbb{Q}}[\phi(S_0, S_1, \dots, S_{T-1}, S_T)] \leq \mathbb{E}_{\mathbb{Q}}[\phi(\underbrace{S_0, S_T, \dots, S_T}_{\text{“early transport”}})]$$

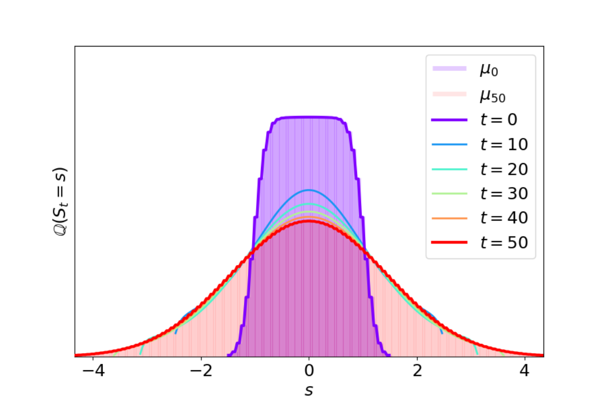
- Late (early) transport solves the lower (upper) bound MOT problem

Late and early transports II

Marginals of the computed optimal solutions:



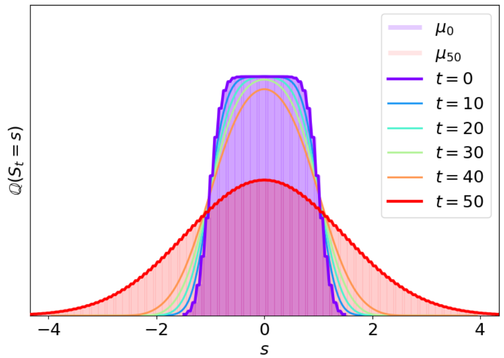
Lower bound problem



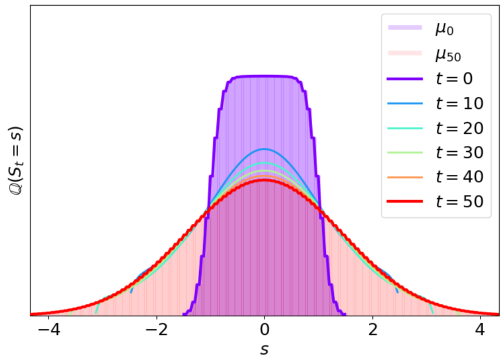
Upper bound problem

Late and early transports II

Marginals of the computed optimal solutions:



Lower bound problem



Upper bound problem

Note that this payoff is of the form $\phi(S_0, \ldots, S_T) = \sum_{t=1}^{T} \phi_t(S_{t-1}, S_t)$ — no process X needed!

- Reduces the size, $(n_S n_X)^{T+1}$, of the problem
- Optimal solutions corresponds to S being Markov under $\mathbb{Q}$

The robust price of an Asian option

- Consider pricing an *Asian straddle* with strike 30,

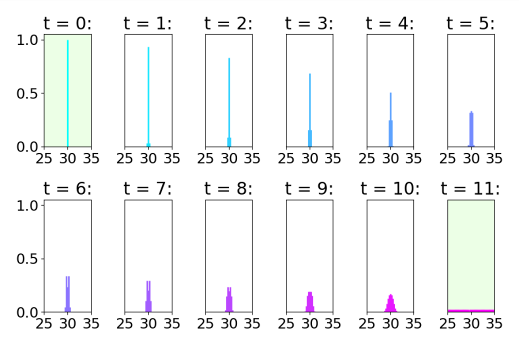
$$\phi(S_0, \ldots, S_T) = |X_T - 30|,$$

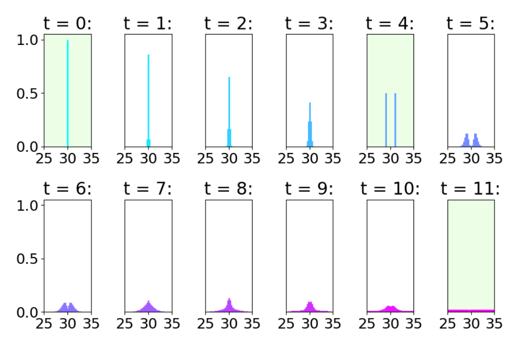
where $X_t := (t+1)^{-1} \sum_{j=0}^{t} S_j, \quad t = 0, \ldots, T,$ is the rolling arithmetic mean.

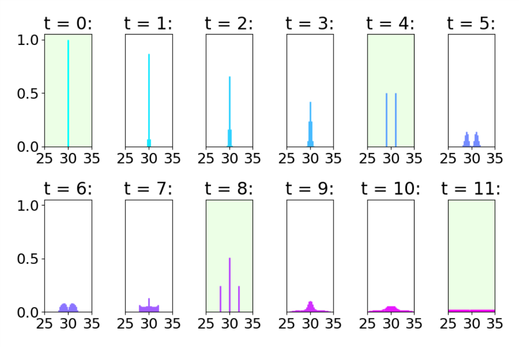
- Optimal solution is
  - ...known when $\mathscr{T} = \{0, T\}$
  - ...conjectured when $\mathscr{T} = \{0, T_0, T\}$ for $0 \leq T_0 \leq T$ (Stebegg 2014)

The robust price of an Asian option II

- The marginals of the computed optimal solution of the lower bound MOT
- Marginals subject to constraints are marked with green







Conclusion

Conclusion

- Accumulation of non-zero errors in the (martingale) constraints?
- Convergence of optimal solutions of the regularized problem as $\varepsilon \to 0$?
- Other ideas...?

Thanks for your attention!
Questions?

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