GMT Exercise Class 21se February.

Question 1. In the proof of Rademacher's Theorem (Theorem 5.1.11), we use that the partial derivatives of f are approximately continuous, but why is this obviously true? We know that every measurable function is approximately continuous, but is it obviously true that the partial derivatives are measurable? Note that even though they are bounded, that alone is not enough to imply measurability (e.g. a function which is 1 on a Vitali set and 0 everywhere else). And did we every prove that the differential is measurable, or is this too considered to be obvious?

Question 2. Could one replace C^1-function in Corollary 5.1.10 with Lipschitz continuous?

Question 3. What is the geometric interpretation of Corollary 5.1.10 (Sard's theorem)?

Question 4. What is a concrete example on how to use the area formula in applications or in ordinary area calculations in multivariable calculus?

Question 5. By introducing Hausdorff measure we could generalise the notion of Jacobian of $f : \mathbb{R}^m \to \mathbb{R}^n$ to any K-dimension. What is the interpretation of the dimension K, for K to be fractional or any number below or beyond m / n?

Question 6. Here is an alternate proof of Rademacher's theorem which I like better than the Krantz version since I find that one quite technical.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex. Any Lipschitz function $f : \Omega \to \mathbb{R}$ is differentiable a.e. in Ω.*

Proof. It is enough to show that f is differentiable in every Lebesgue point of Ω. To this end, fix such an x and introduce for $y \in \overline{B}_1$

$$f_\sigma(y) = \frac{f(x+\sigma y) - f(x)}{\sigma}, \sigma \in (0, \operatorname{dist}(x, \partial\Omega)).$$

Uniform convergence of f_σ to $\langle \nabla f, y \rangle$ would imply that f is differentiable in x, and this is now our goal.

By the Lipschitz continuity of f, we get that f_σ is uniformly equibounded and equicontinuous, whereby the Ascoli-Arzela compactness theorem allows us to simply check whether the limit function $\hat{f}$ of some arbitrary sequence f_{σ_h} coincides with $\langle \nabla f, y \rangle$. (Note that we may consider the sequence itself, and not just the subsequence, for if not then Ascoli-Arzela applied once more to a non-convergent subsequence gives us a contradiction.)

Every Lipschitz function is a Sobolev function in $W^{1,\infty}$, in fact when Ω is bounded, open and convex these spaces are the same.

Thus we have a weak derivative ∇f of f, and so we may estimate

$$\int_{B_1} |\nabla f_{\sigma_h}(y) - \nabla f(x)| dy = \int_{B_1} |\nabla f_{\sigma_h}(x + \sigma y) - \nabla f(x)| dy =$$
$$= \frac{1}{\sigma_h^n} \int_{B_{\sigma_h}(x)} |\nabla f_{\sigma_h}(z) - \nabla f(x)| dz \equiv 0,$$

Letting $h \to \infty$ in the above thus gives us that the weak derivative of our sequence converges in L^1 to the constant $\nabla f(x) \in L^1(\Omega)$. As the sequence itself converges uniformly to $\hat{f}$, it also holds, that the sequence converges in L^1 to $\hat{f}$ since Ω is bounded. All this amount to saying that according to a proposition from Sobolev theory (sorry this part is not so satisfying) we have,

$$f_{\sigma_h} \to^* \hat{f} \text{ and } \nabla \hat{f}(y) = \nabla f(x).$$

By Stokes theorem $\hat{f}(y) = \langle \nabla f(x), y \rangle + c$, an affine function. Again by uniform convergence we get $c = 0$. □

Question 7. I apologise for this question. It looks like another way to look at the k-dimensional Jacobian

$$J_k f(a) = \sup \left\{ \frac{\mathcal{H}^k(Df(a)(P))}{\mathcal{H}^k(P)}; P \text{ is a k-dimensional parallelepiped in } \mathbb{R}^n \right\}$$

in the area and coarea formulae is through so called approximate tangent spaces. Perhaps these formulations are more general, or completely unnecessary for our purposes. Approximate tangent spaces are spaces defined for every point in a $\mathcal{H}^k$-rectifiable set, taking a corresponding role to classical tangent spaces in differential geometry. I introduce some definitions and then I give the formulae in this form.

Let μ be a $\mathbb{R}^n$-valued Radon measure defined on an open set $\Omega \subset \mathbb{R}^m$. Define $\mu_{x,\rho}$ as $\mu_{x,\rho}(B) = \mu(x + \rho B)$ for Borel $B \subset 1/\rho(\Omega - x)$ and $x \in \Omega$. At $x \in \Omega$ we say that μ has approximate tangent space $\pi \subset \mathbb{R}^m$ with multiplicity $\theta \in \mathbb{R}^n$ if $\rho^{-k}\mu_{x,\rho}$ locally weakly* converges to $\theta \mathcal{H}^k \pi$ in $\mathbb{R}^m$ as $\rho \downarrow 0$. Here π is always k-dimensional, and we denote

$$\mathrm{Tan}^k(\mu, x) = \theta \mathcal{H}^k \pi.$$

Intuitively I guess this means that μ measures local sets around x in the same way that the Hausdorff measure measures their projections onto the k-dimensional space π.

Given this notion we can now define the approximate tangent space to a set as follow. For $S \subset \mathbb{R}^m$ a countably $\mathcal{H}^k$-rectifiable set, we define $\mathrm{Tan}^k(S, x)$ to be the approximate tangent space to the measure $\mathcal{H}^k S_i$ at $x \in S_i$ where (S_i) is a partition of $\mathcal{H}^k$-almost all of S into $\mathcal{H}^k$-rectifiable sets.

Let $E \subset \mathbb{R}^m$ be a countably $\mathcal{H}^k$-rectifiable set and $f : \mathbb{R}^m \to \mathbb{R}^n$ Lipschitz. We say that f is tangentially differentiable at $x \in E$ if the restriction of f to the affine space $x+\text{Tan}^k(E,x)$ is differentiable at x. The tangential differential, denoted by $d^E f_x$, is a linear map between $\text{Tan}^k(E,x)$ and $\mathbb{R}^n$.

Let $n \geq k$.

Theorem 2. *Let $E \subset \mathbb{R}^m$ be a countably $\mathcal{H}^k$-rectifiable set and $f : \mathbb{R}^m \to \mathbb{R}^n$ Lipschitz. Then the multiplicity function $\mathcal{H}^0(E \cap f^{-1}(y))$ is $\mathcal{H}^k$-measurable in $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} \mathcal{H}^0(E \cap f^{-1}(y)) d\mathcal{H}^k(y) = \int_E J_k d^E f_x d\mathcal{H}^k(x)$$

Still, $n \geq k$.

Theorem 3. *Let $E \subset \mathbb{R}^m$ be a countably $\mathcal{H}^k$-rectifiable set and $f : \mathbb{R}^m \to \mathbb{R}^k$ Lipschitz. Then the multiplicity function $\mathcal{H}^{n-k}(E \cap f^{-1}(y))$ is $\mathcal{L}^k$-measurable in $\mathbb{R}^k$, $E \cap f^{-1}(y)$ is countably $\mathcal{H}^{N-k}$-rectifiable for $\mathcal{L}^k$-a.e. $y \in \mathbb{R}^k$, and*

$$\int_{\mathbb{R}^k} \mathcal{H}^{n-k}(E \cap f^{-1}(y)) dy = \int_E J_k d^E f_x d\mathcal{H}^n(x)$$

Do we have an expert in the class who has seen these before, what is different here and what have we gained? It seems that should one invest the time in learning these definitions then the proofs look easier (or at least shorter). One remark is that the Area formula in the tangent space version has a Hausdorff measure integration on both equality sides.

Question 8. Lemma 5.1.4 gives an explicit formula for special cases of lower dimensional Jacobians. The proof says it is clear that the parallepiped P, over which we are taking the suprenum, can be chosen in a certain way to make calculations easier. Motivate this choice of P.

Question 9. The proof that Lipschitz functions are differentiable a.e. rely on a induction argument. Prove the base case n=1.

Question 10. Use the area and co-area formulas to prove Corollary 5.1.13. and 5.2.6. Are there any details one should be aware of?

Question 11. Consider $W(\cdot)$ a sample path of a one dimensional Wiener process. Is it true that

$$\limsup_{t \to t_0} \frac{|W(t) - W(t_0)|}{|t - t_0|} = \infty$$

holds for $a.e$ $t_0 \in \mathbb{R}$?

Hint: Can we use the following Theorem (known as Stepanov's Theorem), and also maybe prove it (its proof should follow as a consequence of Rademacher's Theorem)?

Theorem 4. *Let $E \subset \mathbb{R}$ be a measurable set and $f : E \to \mathbb{R}$ a measurable function. Then f is a.e differentiable on*

$$\left\{ x \in E \mid \limsup_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} < \infty \right\}.$$

Consider $W(\cdot)$ a sample path of a one dimensional Wiener process. Is it true that

$$\limsup_{t\to t_0} \frac{|W(t)-W(t_0)|}{|t-t_0|} = \infty$$

holds for $a.e\ t_0 \in \mathbb{R}$?

Question 12.
In the proof of theorem 5.1.9. when estimating the the term $|\langle T, y-z\rangle|$. At one step there is an equality where some extra term $\Pi(f(y)-f(z))$ has been added seemingly out of nowhere. Why can these be added and still have an equality?

Question 13. There is a minor error in the statement of Corollary 5.1.10 (disclaimer: some may classify this as an absurd counter-example and not an actual error.), what is it?

Question 14.[1] The following three theorems are direct consequences of the co-area formula. Explain how:

Theorem 5. [Polar coordinates] *Let $g : \mathbb{R}^n \mapsto \mathbb{R}$ be integrable (w.r.t. the Lebesgue measure). Then*

$$\int_{\mathbb{R}^n} g(x) d\mathcal{L}^n(x) = \int_0^\infty \left(\int_{\partial B_r(0)} g(y) d\mathcal{H}^{n-1}(y) \right) dr.$$

Theorem 6. [Levelset integration] *Assume that $f : \mathbb{R}^n \mapsto \mathbb{R}$ is Lipschitz. Then*

$$\int_{\mathbb{R}^n} |Df(x)| d\mathcal{L}^n(x) = \int_{-\infty}^\infty \mathcal{H}^{n-1}(\{x \in \mathbb{R}^n;\ f(x) = t\}) dt.$$

Theorem 7. [Levelset integration again] *Assume that $f : \mathbb{R}^n \mapsto \mathbb{R}$ is Lipschitz with $|Df(x)| > \delta > 0$ a.e. (the value of $\delta > 0$ doesn't matter). Also assume that $g : \mathbb{R}^n \mapsto \mathbb{R}$ is integrable. Then*

$$\int_{\{x\in\mathbb{R}^n;\ f(x)>t\}} g(x) d\mathcal{L}^n(x) = \int_t^\infty \left(\int_{\{x\in\mathbb{R}^n;\ f(x)=s\}} \frac{g(x)}{|Df(x)|} d\mathcal{H}^{n-1} \right) ds.$$

[1] These observations are taken from Evans-Gariepy's "Measure theory and fine properties of fuctions". I think that the examples are good and should be mentioned in relation to the rather technical formulation of the co-area formula.